MATHEMATICAL MODELING AND DESIGN

Luigi Mussio [(1)] – Giovanna Sona [(2)]
[(1)] Politecnico di Milano – DICA – Piazza L. da Vinci, 32 – 20133 Milano
Tel. 02-2399-6501 – Fax 02-2399-6602 – e-mail luigi.mussio@polimi.it
[(2)] Politecnico di Milano – DICA – Piazza L. da Vinci, 32 – 20133 Milano
Tel. 02-2399-6530 – Fax 02-2399-6550 – e-mail giovanna.sona@polimi.it

Abstract – The main goal of GIS is the representation of reality, using models that can represent it the best they can. In this way, every further analysis and representation can be inserted in a proper geographic cloud. Indeed often we can have different interpretations of the same phenomenon and we don't have a unique solution for the creation of the model. The different interpretation of the phenomena often depends on the person who observes it, and on all the different persons involved in the modeling process.

Each of them could have a different way to interpret and describe what they see. It is not always so easy to create a model of reality, for example deciding where is the beginning of a forest or a lake depends on the perspective with which the model's designer observes the phenomenon.

We can have also time-depending objects that cannot have obviously a unique and defined interpretation, but it will depend on the instant of observation. Above all the representation of reality depends on the detail level we have to describe and from the future use of the model. The representation of reality:

- depends on the perspective of the model's designer;
- prescribes what users will be able to do when extracting information;
- is very difficult (or often impossible) to be transformed at a larger stage;
- is seldom for multiple purposes.

This subjectivity of representation obliges to make a formalization of abstraction processes of reality, in a way to set an efficient and consistent object able to interpret geographical information.

Successively a method for automatic reconstruction of models of a piecewise smooth surface is illustrated.

- Very often, we have a set of positions and want a curve to interpolate (pass through) them smoothly; so the Catmull-Rom family of interpolating or approximating splines, also called Over-hauser splines, are useful in this situation.
- When the object has a complex shape, it is very difficult to obtain a good reconstruction and a real-looking representation. If the object has a continuos curvature (convex hull) it is possible to generate a 3D triangular surface employing the Delaunay algorithm, generalized to the 3D case.
- Then if one wants to have a good result in terms of visual display it is necessary to apply another algorithm that produces Bézier splines on a triangular support. Of course, a realistic reconstruction of the object shape needs a high hardware performance to obtain as a final product the display of the interpolated surface (using CAD software).

Some applications will be shown by means of examples, wich present both GIS implementation and graphic computer aided representation.

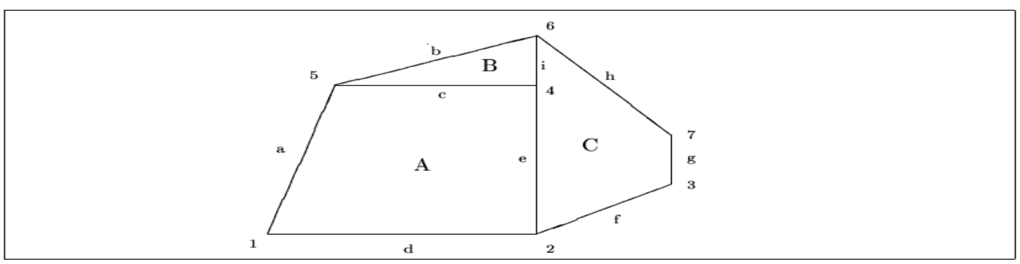
PART I – GIS

Geographic Information System

Project steps

- external, descriptive and completely free;
- conceptual, descriptive, but formal;
- logical, implemented by suitable structure, as:
  - a table;
  - a tree (in hierarchical form);
  - a network;
  - in relational form (using dual properties).
- physical (by software in the computer).

A vector example



AREA	
ID_AREA	name
I	

SURFACE		
ID_surf.	area	type
A		
B		
C		

LINE	
ID_line	length
a	
b	
c	
d	
e	
f	
g	
h	
i	

POINT		
ID_point	X	Y
1	0.	0.
2	4.	0.
3	6.	1.
4	4.	3.
5	1.	3.
6	4.	4.
7	6.	2.

- Implemented by a tree (in hierarchical form);

ID_AREA	ID_surf.
I	A
I	B
I	C

ID_area	ID_line	ID_ initial point	ID_final point
A	a	1	5
A	c	5	4
A	e	4	2
A	d	2	1
B	b	5	6
B	i	6	4
B	c	4	5
C	e	2	4
C	i	4	6
C	h	6	7
C	g	7	3
C	f	3	2

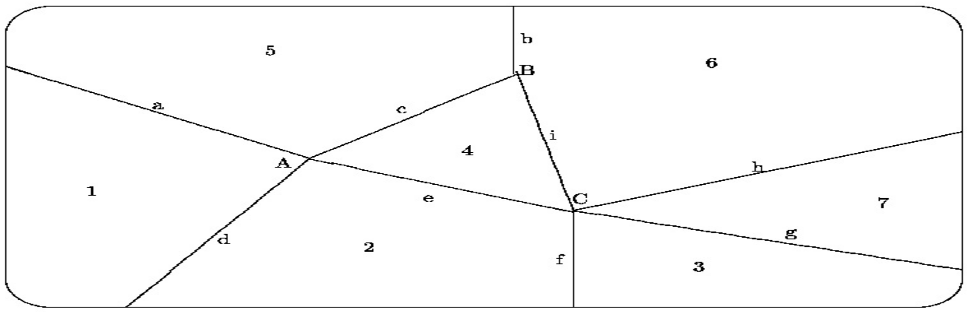
- implemented by a network;

ID_area	ID_line
A	a
A	c
A	d
A	e
B	b
B	c
B	i
C	e
C	f
C	g
C	h
C	i

ID_line	ID_initial point	ID_final point
a	1	5
b	5	6
c	4	5
d	1	2
e	2	4
f	2	3
g	3	7
h	6	7
i	4	6

- its corresponding dual tree, using dual properties (again in hierarchical form), where:
    - surfaces become nodes;

- points become regions;
- line remain line (often called arcs).



ID_space	ID_region
S	1
S	2
S	3
S	4
S	5
S	6
S	7

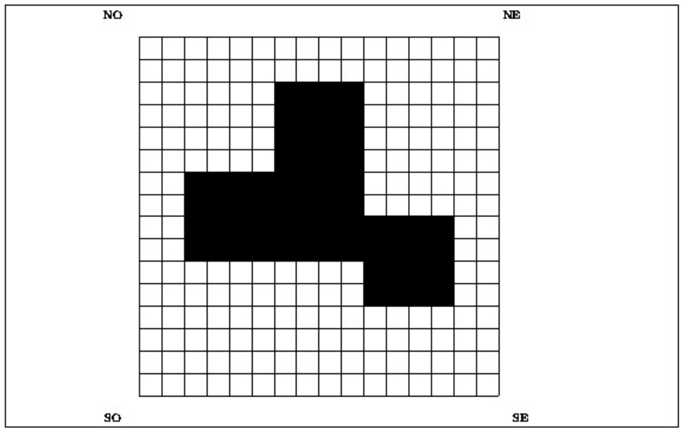
ID_region	ID_arc	ID_region	ID_arc	ID_region	ID_arc
1	a	3	g	5	external
1	d	3	external	6	b
1	external	4	c	6	h
2	d	4	e	6	i
2	e	4	i	6	external
2	f	5	a	7	g
2	externak	5	b	7	h
3	f	5	c	7	external

- implemented in relational form (using dual properties).

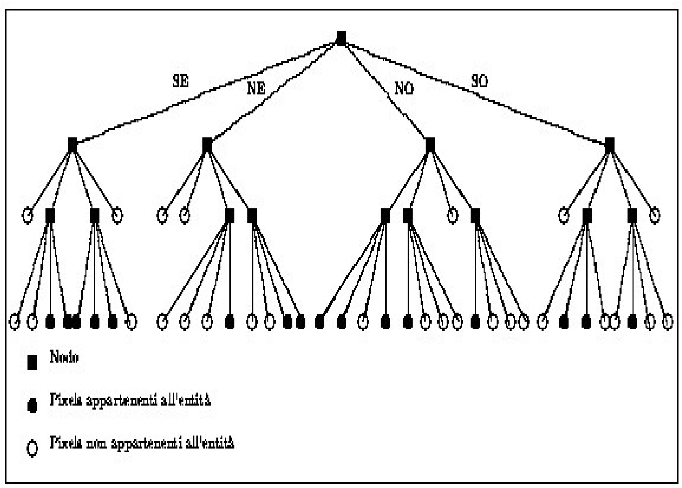
ID_line	ID_initial point	ID_final point	ID_right area	ID_left area
a	1	5	A	external
b	5	6	B	external
c	4	5	B	A
d	2	1	A	external
e	2	4	C	A
f	3	2	C	external
g	7	3	C	esterno
h	6	7	C	esterno
i	4	6	C	B

A raster example

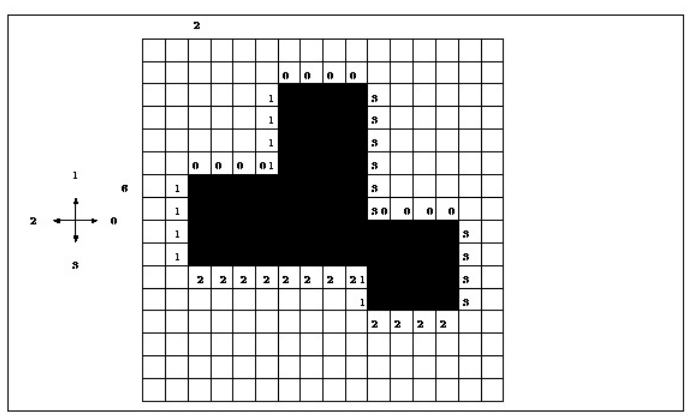
Organization rules (in decreasing order): dominance, importance and center.



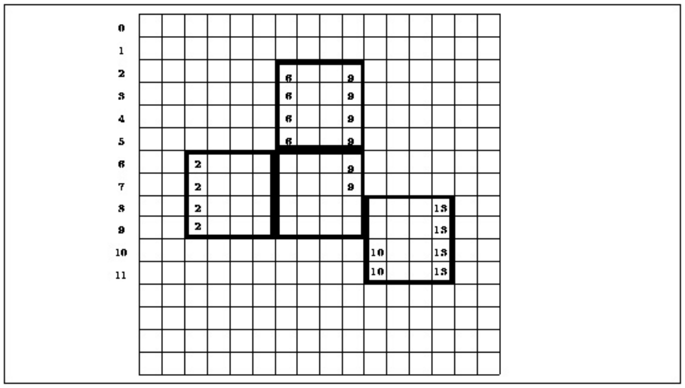
Compression tecniques:



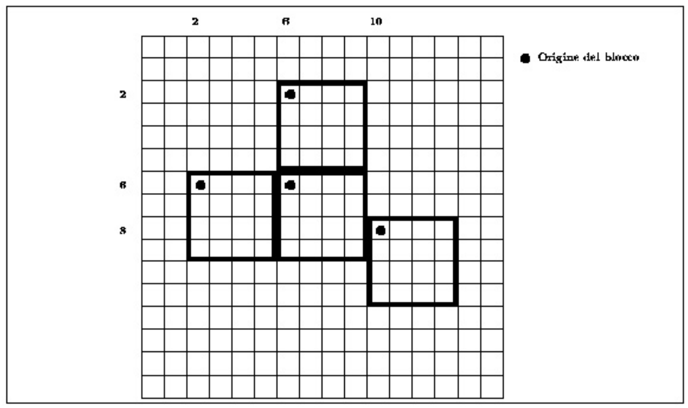
(Pyramid) quadtres



Chain codes



Run-lengh codes



Block codes

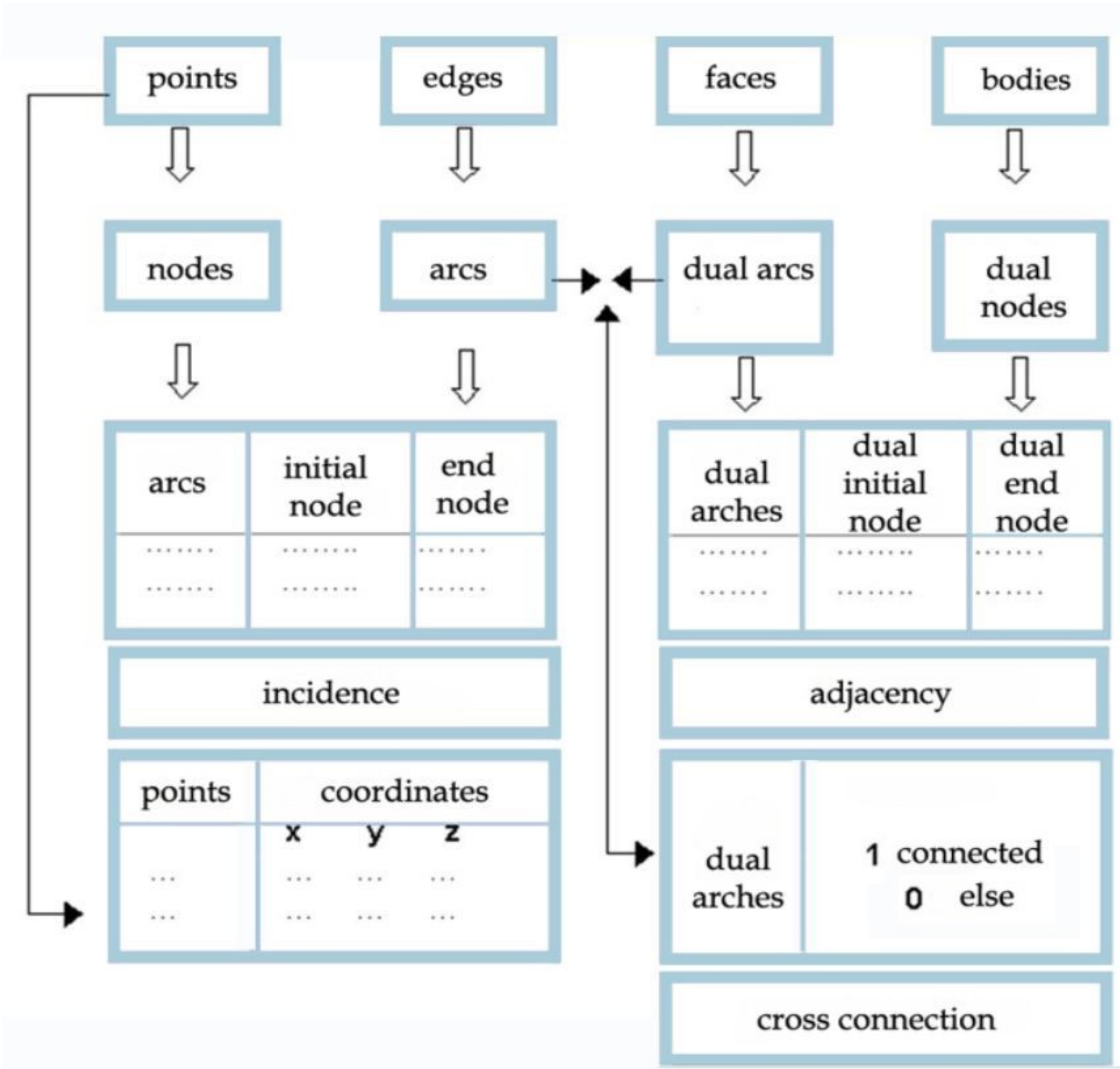
RUN-LENGTH CODES	
Row	Column
2	6, 9
3	6, 9
4	6, 9
5	6, 9
6	2, 9
7	2, 9
8	2, 13
9	2, 13
10	10, 13
11	10, 13

BLOCK CODES	
Origin	Arc
6,2	4
2,6	4
6,6	4
8,10	4

A 3D vector example

- A point connects more than one arc, polygon and polyhedron;

- a line connects only two points, but more than one polygon and polyhedron;
- a polygon connects more than one point and line, but only two polyhedrons;
- a polyhedron connects more than one point, line and polygon.



- with the corresponding dual [1] tree, using dual properties associates, which associates:

  - polydrons to nodes (i.e. dual points);
  - polygons to arcs (i.e. dual lines);

  so that a "crosslink" table joins the incidence table (between points and lines) and the adjacency table (between polygons and polyhedrons).

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[1] Duality is a property, which permits to associate two different classes of objects, passing the characteristics of the first class to the second one. The generalization of duality is the complmentary, which allows for linking the features of many classes together.

A 3D raster example

- (Pyramid) octries (as for image and/or map pyramids);
- generalized run-lengh codes (sequentially plane by plane)
- (solid) block codes.

Time-variant Dynamic GIS and its characteristic time

- Constant (i.e. non time-variant);
- periodic or quasi-periodic;
- a–periodic (i.e. with a drift);
- in mixed form;
- with one or more breakdown point(s).

Object modeling

- 3 classes of relations in the monodimensional space: between points, points and lines, lines;
- 6 classes of relations in the bidimensional space: beyond to those of the previous case, between points and surfaces, lines and surfaces, surfaces;
- 10 classes of relations in the three-dimensional space: beyond to those of the previous cases, between points and 3D bodies, lines and 3D bodies, surfaces and 3D bodies, 3D bodies.

<u>In one dimension, there exist 7 topological relations</u>

- Point – point: separate
coincident
- Point – line: external
internal
- Line – line: separate
connected
internal

<u>In two dimensions, 10 new relations join to the previous ones,
defined for the monodimensional case, reaching the number of 17 topological relations</u>

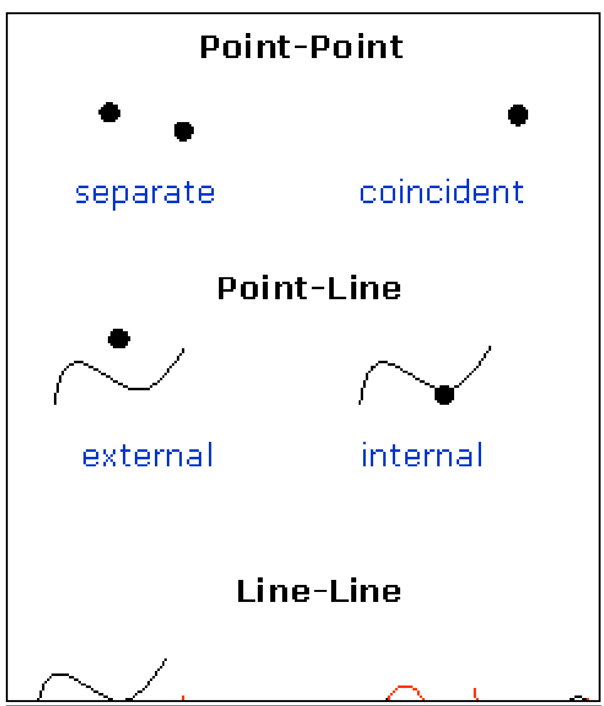
- Point – surface: external
internal
- Line – surface: external
connected
secant
internal
enucleating

- Surface – surface: external
  connected
  internal

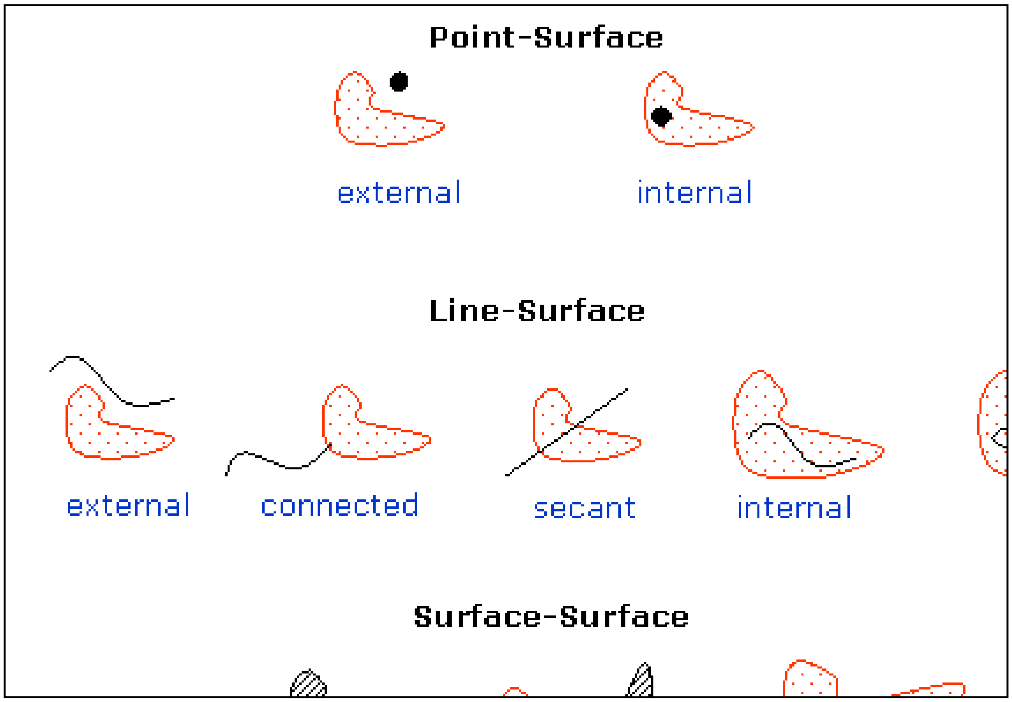
<u>In three dimensions, 15 new relations join the previous ones,
defined for the bidimensional case, reaching the number of 32 topological relations</u>

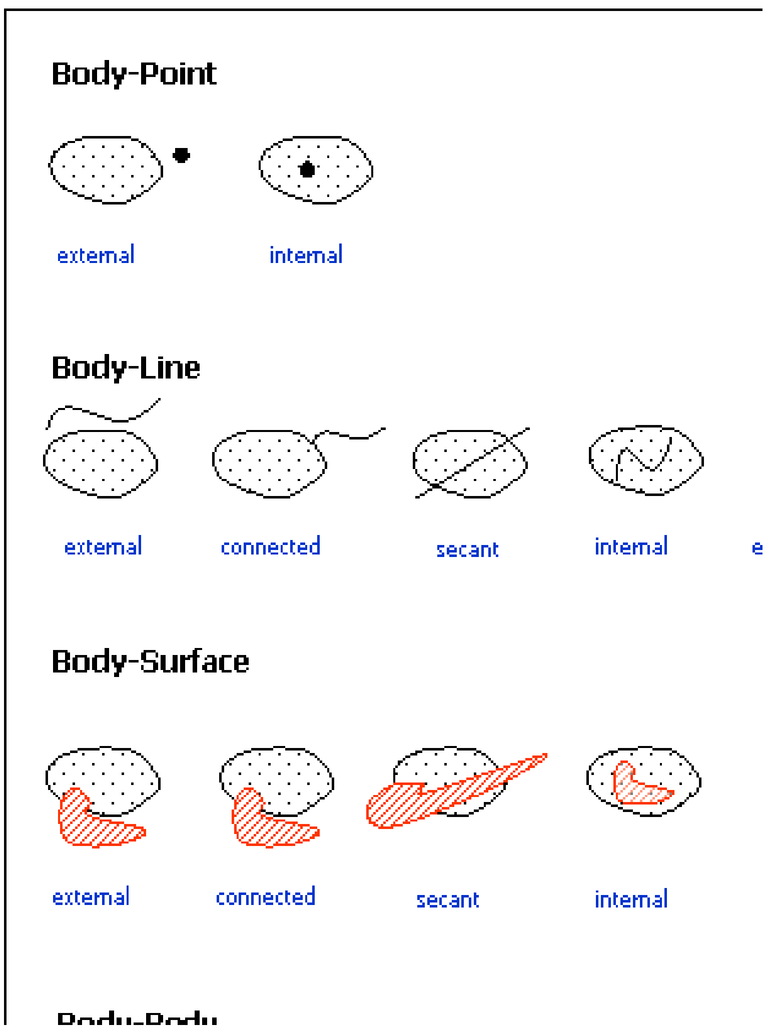
- Point – body: external
  internal
- Line – body: external
  connected
  secant
  internal
  enucleating
- Surface – body: external
  connected
  secant
  internal
  enucleating
- Body – body: external
  connected
  internal

Topological Relations in 1D



Topological Relations in 2D





Topological Relations in 3D

Main geometric relations among the primary elements

- 10 geometric relations, for the mono-dimensional case;
- 32 geometric relations, for the two-dimensional case;
- 230 geometric relations, for the three-dimensional case.

Mono-dimensional case:

❑ Point – point:	separate	(1)	
	coincident	(2)	
❑ Point – line:	external	(3)	
	marginal	(4)	
❑	internal	(5)	
❑ Line – line:	external	(6)	
	semi-external	(7)	
	overlapping	(8)	
	semi-internal	(9)	
	internal	(10)	

Bidimensional case:

❑ Point – point:	separate	(1)	
	coincident	(2)	
❑ Point – line:	external	(3)	
	marginal	(4)	only if the line is open
	internal	(5)	
	included	(6)	only if the line is close
❑ Point – surface:	external	(7)	
	marginal	(8)	
	internal	(9)	
❑ Line – line:	external	(10)	
	semi-external	(11)	
	semi-intersect.	(12)	
	intersecting	(13)	
	overlapping	(14)	
	semi-internal	(15)	
	internal	(16)	
	semi-included	(17)	only at least a line is close
	included	(18)	
❑ Line – surface:	external	(19)	
	semi-external**Errore. Il segnalibro non è definito.**	(20)	
	semi-marginal	(21)	
	marginal	(22)	

intersecting (23)
secant (24)
semi-internal (25)
internal (26)
enucleating (27)

- Surface – surface: external (28)
  semi-external (29)
  overlapping (30)
  semi-internal (31)
  internal (32)

Three-dimensional case

<u>Note:</u> In this case, the classification of the geometric relations allows various possibilities. In most cases considering the number of the geometric relations in correspondence to the cardinality of the group of spatial symmetry, it is quite easy to reach the number of 230 elements. In fact, while a point is a point and a body a body, a line can be open or close, and it can occupy an open area or close area, as well as a volume; moreover a surface can be open or close, and it can occupy a volume. Therefore five distinguished types of lines are taken into account: stick, ring, doily, basket and skein, and three distinguished types of superficial are also taken into account: leaf, ball and bag. The resulting connections are:

- Point – point: separate (1)
  coincident (2)
- Point – stick: separate (3)
  marginal (4)
  internal (5)
- Point – ring: separate (6)
  internal (7)
- Point – doily: separate (8)
  marginal (9)
  internal (10)
- Point – leaf: separate (11)
  marginal (12)
  internal (13)
- Point – basket: separate (14)
  internal (15)
  included (16)
- Point – ball: separate (17)
  internal (18)
  included (19)
- Point – skein: separate (20)
  marginal (21)
  internal (22)
- Point – bag: separate (23)
  marginal (24)
  internal (25)
- Point – body: separate (26)
  marginal (27)
  internal (28)
- Stick – stick: separate (29)
  marginal (30)
  intersected (31)
  internal (32)
- Stick – ring: separate (33)
  marginal (34)
  intersected (35)
  internal (36)
- Stick – doily: separate (37)
  marginal (38)
  overlapping (39)

marginal (40)
internal (41)

- Stick – leaf: separate (42)
  marginal (43)
  overlapping (44)
  marginal (45)
  internal (46)
- Stick – basket: separate (47)
  marginal (48)
  overlapping (49)
  marginal (50)
  included (51)
- Stick – ball: separate (52)
  marginal (53)
  overlapping (54)
  marginal (55)
  included (56)
- Stick – skein: separate (57)
  marginal (58)
  overlapping (59)
  marginal (60)
  internal (61)
- Stick – bag: separate (62)
  marginal (63)
  overlapping (64)
  marginal (65)
  internal (66)
- Stick – body: separate (67)
  marginal (68)
  overlapping (69)
  marginal (70)
  internal (71)

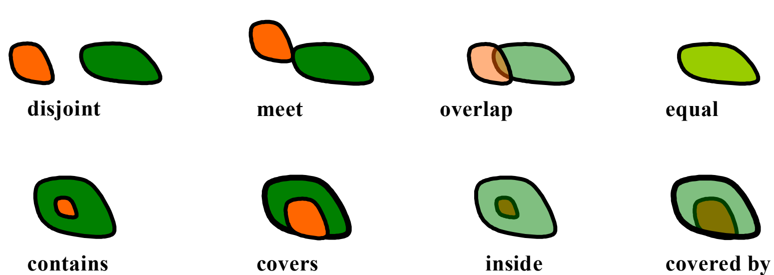
- Ring – ring: separate (72)
  tangent (73)
  linked (74)
  intersected (75)
- Ring – doily: separate (76)
  tangent (77)
  overlapping (78)
  internal (79)
- Ring – leaf: separate (80)
  tangent (81)
  overlapping (82)
  internal (83)
- Ring – basket: separate (84)
  tangent (85)
  overlapping (86)
  marginal (87)
  included (88)
- Ring – ball: separate (89)
  tangent (90)
  overlapping (91)
  marginal (92)
  included (93)
- Ring – skein: separate (94)
  tangent (95)
  overlapping (96)
  marginal (97)
  internal (98)
- Ring – bag: separate (99)
  tangent (100)
  overlapping (01)
  marginal (02)
  internal (03)
- Ring – body: separate (04)
  tangent (05)
  overlapping (06)
  marginal (07)
  internal (08)

- Doily – doily: separate (09)
  tangent (10)
  overlapping (11)
  internal (12)
- Doily – leaf: separate (13)
  tangent (14)
  overlapping (15)
  internal (16)
- Doily – basket: separate (17)
  tangent (18)
  overlapping (19)

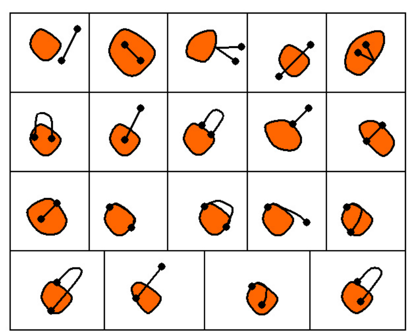
	marginal	(20)
	included	(21)
❑ Doily – ball:	separate	(22)
	tangent	(23)
	overlapping	(24)
	marginal	(25)
	included	(26)
❑ Doily – skein:	separate	(27)
	tangent	(28)
	overlapping	(29)
	marginal	(30)
	internal	(31)
❑ Doily – bag:	separate	(32)
	tangent	(33)
	overlapping	(34)
	marginal	(35)
	internal	(36)
❑ Doily – body:	separate	(37)
	tangent	(38)
	overlapping	(39)
	marginal	(40)
	internal	(41)
❑ Leaf – leaf:	separate	(42)
	tangent	(43)
	overlapping	(44)
	internal	(45)
❑ Leaf – basket:	separate	(46)
	tangent	(47)
	overlapping	(48)
	marginal	(49)
	included	(50)
❑ Leaf – ball:	separate	(51)
	tangent	(52)
	overlapping	(53)
	marginal	(54)
	included	(55)
❑ Leaf – skein:	separate	(56)
	tangent	(57)
	overlapping	(58)
	marginal	(59)
	internal	(60)
❑ Leaf – bag:	separate	(61)
	tangent	(62)
	overlapping	(63)
	marginal	(64)
	internal	(65)
❑ Leaf – body:	separate	(66)
	tangent	(67)
	overlapping	(68)
	marginal	(69)
	internal	(70)
❑ Basket – basket:	separate	(71)
	tangent	(72)
	overlapping	(73)
	included	(74)
❑ Basket – ball:	separate	(75)
	tangent	(76)
	overlapping	(77)
	included	(78)
❑ Basket – skein:	separate	(79)
	tangent	(80)
	overlapping	(81)
	incl. / internal	(82)
❑ Basket – bag:	separate	(83)
	Tangent	(84)
	overlapping	(85)
	incl. / internal	(86)
❑ Basket – body:	separate	(87)
	tangent	(88)
	overlapping	(89)
	incl. / internal	(90)
❑ Ball – ball:	separate	(91)
	tangent	(92)
	overlapping	(93)
	included	(94)
❑ Ball – skein:	separate	(95)
	tangent	(96)
	overlapping	(97)
	incl. / internal	(98)

- Ball – bag: separate (99)
  tangent (200)
  overlapping (01)
  incl. / internal (02)
- Ball – body: separate (03)
  tangent (04)
  overlapping (05)
  incl. / internal (06)
- Skein – skein: separate (07)
  tangent (08)
  overlapping (09)
  internal (10)
- Skein – bag: separate (11)
  tangent (12)
  overlapping (13)
  internal (14)
- Skein – body: separate (15)
  tangent (16)
  overlapping (17)
  internal (18)
- Bag – bag: separate (19)
  tangent (20)
  overlapping (21)
  internal (22)
- Bag – body: separate (23)
  tangent (24)
  overlapping (25)
  internal (26)
- Body – body: separate (27)
  tangent (28)
  overlapping (29)
  internal (230)

Note: This long list could further be lengthened, taking into account particular conditions of tangency, intersection and superimposition. Furthermore these conditions can be simple, double or multiple and can be punctual, linear and aerial. Therefore taking into account the length of the present list and the richness of the proposed under-classifications (as shown in the following figures), maybe also 4783 cases could be found according to the cardinality of the recently discovered four-dimensional symmetry group.



Region-Region Relations



Line-Region Relations

Disjoint	Contains	ve·consider·a· ne·intersectior Bypass
Meet	Covered	Fork
Equal	Inside	
Overlap	Covered by	

Line-Line Relations

The problem

Several applications in data imaging and modeling require the definition of conceptual models, which imply the knowledge of the data topology. Codification of topology is really important to avoid different interpretations of the same object or phenomenon:

- a classical approach lists all Boolean relations and selects among them the admissible ones from a geometric point of view;
- an alternative approach combines the topological and geometric relations between primary elements with the symmetries (in the spaces in which the complex objects are located).

The alternative has been recently developed; herefore it will be here presented in order to remark analogies and differences, enhancing how to treat common data independent from a reference frame.
Object modeling allows the widest modalities to represent bodies and features for their studies and analysis. The relations between characteristic elements (points, lines, surfaces and 3D bodies) determine the validity of this modeling and their sets have correspondences with the groups of symmetry in the mono-dimensional, two-dimensional and three-dimensional spaces of the formal algebra.
The object (eventually dynamic, because time-variant) modeling is an innovative and technologically advanced instrument for the study of spatial phenomena and their temporal dynamics. Unfortunately the use of this instrument is still rare; in fact it remarkably increases the complexity of realization and management of the informatics systems.
The production of the cartography has been, for a long time, a refined art. The maps, created from geodetic measurements and updated thickening by survey operations, are completed from the handicraft experience of the mapmakers.
The impressive spread of technology is carrying to give over the traditional map production systems, preferring the new and powerful digital systems of map production.
The object paradigm supplies a high level of abstraction of the physical structure of the data; it is able to model the system according to the definition of the data (objects), as well as to operate on the concept of attributes and relations, being both stored as an integrating part of the same objects.
The object oriented representation permits to create complex objects, like polygons, 3D bodies, etc, to analyze and to manipulate them like single objects (even if they are combinations of objects).
This characteristic eliminates the necessity of clarifying all the geographic and semantic attributes of the objects. Furthermore the object-oriented approach allows the easy description of new types of data, defining operations on new objects and structuring the objects in a hierarchical way. In this context it is of particular interest to use those tools, like the extensions, able to explore the data, to process them in external phases where specific operations are executed, and to import the results; because of their high qualification, the dimension and complexity of these external procedures exclude their insertion inside the system.
The development of object oriented systems is going in two different directions: adding all the typical functions of the object paradigm to a relational database, or constructing a new independent system. The first approach is very robust, because the databases are a mature product, while the second approach is still an open issue in the scientific and technological research. In both cases, the main objective is the production of a set of procedures able to process spatially referenced data.

The most important procedures for the data management tend to solve these main problems: data storage and information retrieval according to suitable requirements. Particularly for the management of spatially referenced data it is necessary to create new entities (polygons, 3D bodies, etc.) and to define rules for their processing, like distance, incidence, adjacency, etc. However in the object oriented systems, some problems with difficult solution, presented by the relational models, are still not completely solved. At present time, a hybrid system seems to be the best solution for the creation of robust physical (relational) data structures and flexible logical (object oriented) data models.

An object can be described as a conceptual entity, easily defined by its data and environment. The environment includes a set of operations and methods, valid on the object itself. Its state is represented by the values assumed by local variables. Every single object belongs to a class, which defines the type of object. The classes can have variables, describing the characteristics of the class itself. If some classes have common variables and methods, it is possible to define a super–class, which groups the variables and the methods of all these classes. For convention, the universal super–class (i.e. the root of the system) constitutes the first level of the hierarchical level structure describing the system.

An interesting object oriented data structure can be obtained by introducing a linkage between spatially referenced objects and their features, for example areas with their boundaries, edges with isolated vertices. Thus the spatially referenced objects are identified and described by means of their geometric and thematic features. This observation leads to a first formal requirement, in order to construct a formal data structure:

- every object must be associated to an identifier (name or number);
- every identifier must have a link to the attributes of an object.

Objects with common geometric or thematic aspects can be grouped into the same class (set of objects), identified by a certain name; the list of attributes of a class supplies the names of the attributes, whose values are assumed by the objects themselves. This means that every object, belonging to a class, has a list of values, one for each attribute of the class.

Common attributes lead to a super–class (i.e. a class of classes), which is associated to the list of attribute values assumed by the classes. It is evident that the construction of a super-class introduces a hierarchy in the level structure of the classes.

At the top level, the super-class presents its list of attributes and methods. At the intermediate level, the class presents a list of attributes subdivided in two parts: the former contains the values of the attributes associated to the super–class, the latter contains the list of attributes of the class itself. At the lowest level, the object presents a list of attributes estimated by the lists of the attributes of the super–class or the belonging class. The estimation of an attribute, at the class level, implies that all the objects belonging to the same class have the same value for the selected attribute.

Furthermore it is possible to introduce more hierarchical levels: every level inherits the list of attributes of the higher level and transmits the list of attributes to the lower level. At the lowest level of the hierarchy, all the attributes must be estimated.

Classes of objects and objects are defined so that every object belongs to one (and only one) class; this requirement is equivalent to the convention according the classes to which must be mutually exclusive.

The objects can be classified according to their geometric or thematic features; therefore a second convention states that a class contains only objects of a certain type.
Sometimes this convention seems to be too rigid; however it allows the construction of a very simple and transparent structure. In some rare cases, a possible solution is the construction of complex objects.
The last convention implies a relation many-to-one type between objects and classes. Using this object-oriented approach, geometric structures with different complexity can be described by objects belonging to suitable classes of features. For instance:

- a punctual structure, by means of an object belonging to the class of the points;
- a liner structure, by means of an object belonging to the class of the lines;
- a bidimensional structure, by means of an object belonging to the class of the surfaces;
- a three-dimensional structure, by means of an object belonging to the class of the 3D bodies.

In order to construct a formal data structure, it is necessary to identify the geometric features of the object and their relations. This can be done with different methods. A possible approach uses the graph theory, putting into a one-to-one relation the thematic and the corresponding geometric elements, and describing the topological relations between the various elements. The elements can be described in two different ways:

- in a parametric form, i.e. through parameters, which define the equation of a mathematical curve;
- through a sequence of points connected by a polyline, being the link between two consecutive points a segment of one straight line.

The second solution implies the introduction of new types of points beyond the nodes: these new points only contain information concerning the point positions. In both cases, the following conventions are adopted:

- when a complex object is analyzed through a graph, all the points describing its geometry are treated as nodes, each node having the position given by the coordinates of the corresponding point;
- by using the duality principle, all the planar figures derived by the decomposition of a complex figure are treated like nodes of a dual graph, each node having the position given by the coordinates of the centroids of the same figure;
- by using the duality principle, the 3D bodies, derived by the decomposition of a complex solid object, are treated like nodes of a dual graph, each node having the position given by the coordinates of the centroids of the same body;
- all the segments of straight lines are represented by edges of the graph and each edge has an initial and a final node;
- by using the duality principle, all the surfaces in 3D space are represented by edges of a dual graph and each edge has an initial and a final node.

In order to avoid geometric ambiguities, two new conventions are introduced:

- for each couple of nodes, there is no more than one edge, connecting them;

- the edges cannot be intersected in the simple case of a planar graph; the same requirement is imposed for the entire planar sub-graphs, derived by a possible decomposition of a spatial graph.

The connection between geometric and thematic elements is performed through identifiers. The successive step, in the definition of a formal data structure for a complex object, consists of the analysis of the linkage between the geometric elements (nodes, edges) and the thematic elements (points, lines, surfaces and 3D bodies):

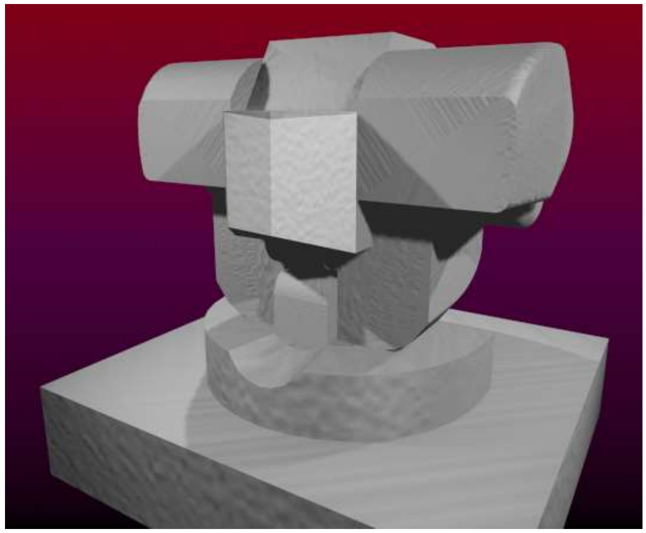
- the linkage between features, like points (thematic elements) and nodes (geometric elements) consists of the following condition: every point is represented by one node (and one only);
- if one node does not represent any feature, a null identifier is used.

The connection between the other geometric and thematic elements with more complexity is set up as follows:

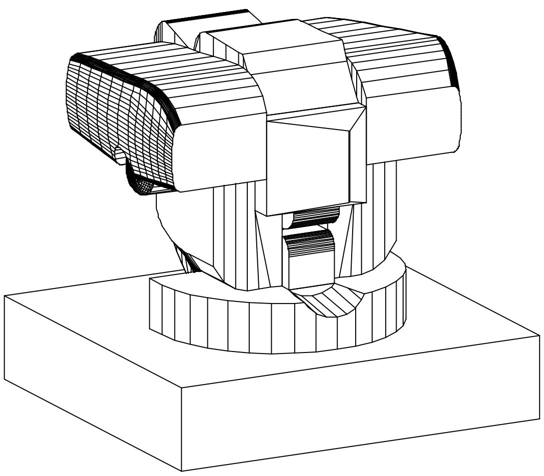
- an edge can be part of a characteristic line;
- if an edge does not belong to a characteristic line, a null identifier is imposed;
- in the simple case (planar graph or planar subgraphs derived by a decomposition of a spatial graph) an edge has always one (and only one) characteristic area at its right and one (and only one) characteristic area at its left;
- a face (i.e. an edge, by using the duality principle) can be part of a characteristic surface (i.e. a characteristic line, by using the same duality principle);
- if a face does not belong to a characteristic surface, a null identifier is imposed;
- by using the duality principle, a surface in 3D space has always one (and only one) characteristic 3D body at its right and a characteristic 3D body 3D (and only one) at its left;
- if an edge (or a face) is a boundary element, one of the two characteristic surfaces (or one of the two characteristic 3D bodies) is called external element;
- if a characteristic line (or a characteristic surface) is a boundary element between two characteristic surfaces (or two characteristic 3D bodies), the edges (or the faces) belonging to this boundary are part of the same characteristic line (of the same characteristic surface);
- on the contrary, if an edge (or a face) is not part of a characteristic line (or a characteristic surface), or it does not belong to a certain boundary, the edge and its characteristic line (or the face and its characteristic surface) intersect a characteristic surface (or a characteristic 3D body) and the right-left linkage is referred to the same surface (or to the same 3D body).

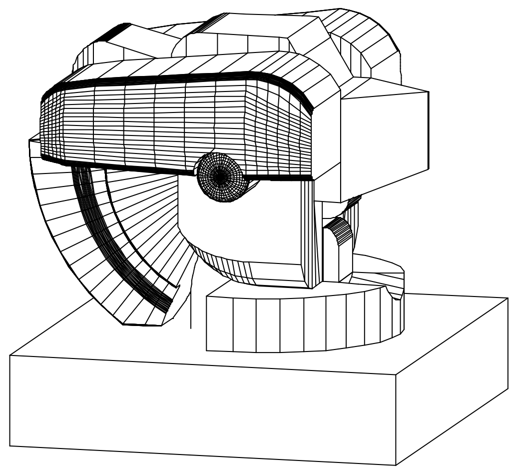
Finally in the case of 3D bodies modeling, it is necessary to establish a cross-connection table between edges and faces (or characteristic lines and characteristic surfaces), so that a topological linkage between primary graphs and dual graphs is present.
In fact, while this linkage is directly defined by the edges in the planar graphs, the primary and dual spatial graphs are completely separated (if they are not previously decomposed in planar sub-graphs); as a consequence, the above mentioned cross-connection table is strictly required.

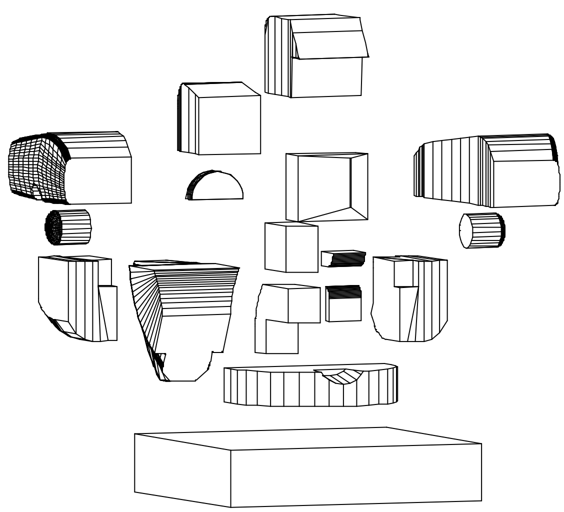
Three real examples



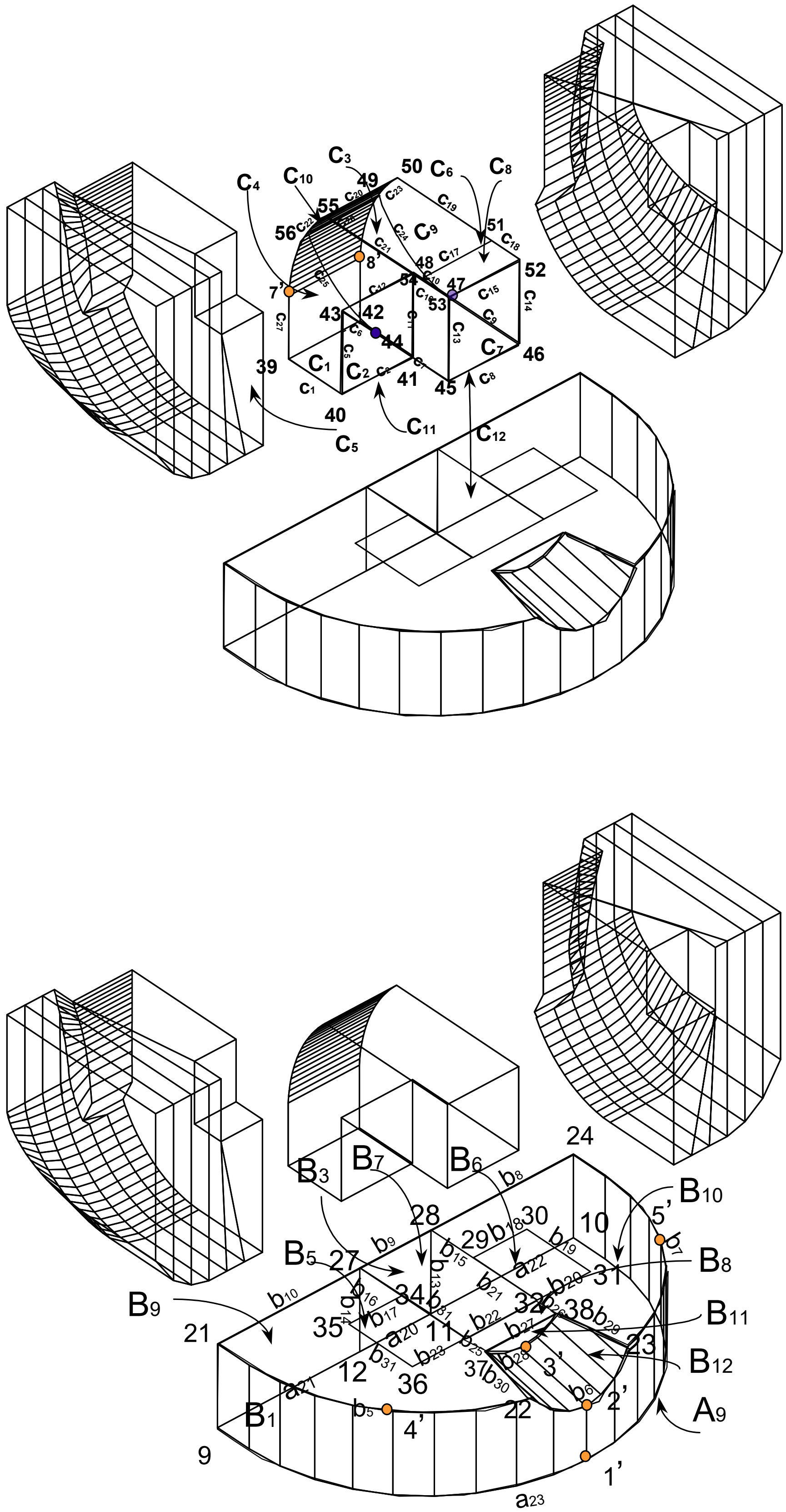
Leonardo da Vinci square fountain (Milan)







Solid modeling and element explosion

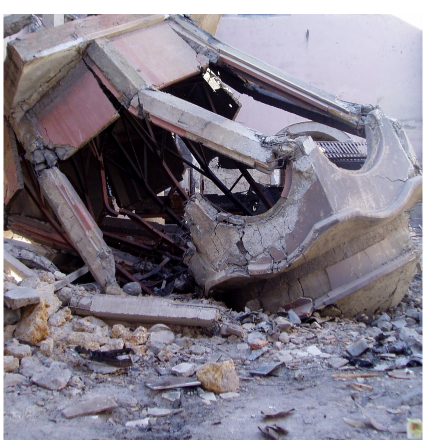


A vector reconstruction

Mušutište (Kosovo) church before and after il 2004

Fresco in the apse



Detail of the distruction

CHARACTERISTIC POINTS	COORDINATES		
	x	y	z
1	0.000	2.000	0.000
6	2.000	2.000	0.000
13	2.000	2.000	1.000
14	2.000	2.000	1.200
3	10.000	2.000	0.000
9	10.000	2.000	1.000
10	10.000	2.000	1.200
2	12.000	2.000	0.000
5	2.000	11.000	0.000
12	2.000	11.000	1.000
4	10.000	11.000	0.000
11	10.000	11.000	1.000
7	2.000	0.000	0.500
15	2.000	0.000	1.200
8	10.000	0.000	0.500
16	10.000	0.000	1.200
21	3.000	2.000	1.000
22	3.000	2.000	1.200
17	9.000	2.000	1.000
18	9.000	2.000	1.200
20	3.000	10.000	1.000
19	9.000	10.000	1.000
23	4.000	2.000	1.200
24	5.000	1.000	1.200
25	7.000	1.000	1.200
26	8.000	2.000	1.200
30	3.000	2.000	5.000
29	3.000	10.000	5.000
27	9.000	2.000	5.000
28	9.000	10.000	5.000
32	6.000	2.000	6.000
31	6.000	10.000	6.000
35	6.000	2.000	4.500
33	5.000	2.000	5.500
34	7.000	2.000	5.500
36	4.000	2.000	4.000
39	8.000	2.000	4.000
37	5.000	1.000	4.000
38	7.000	1.000	4.000
40	6.000	12.000	0.000

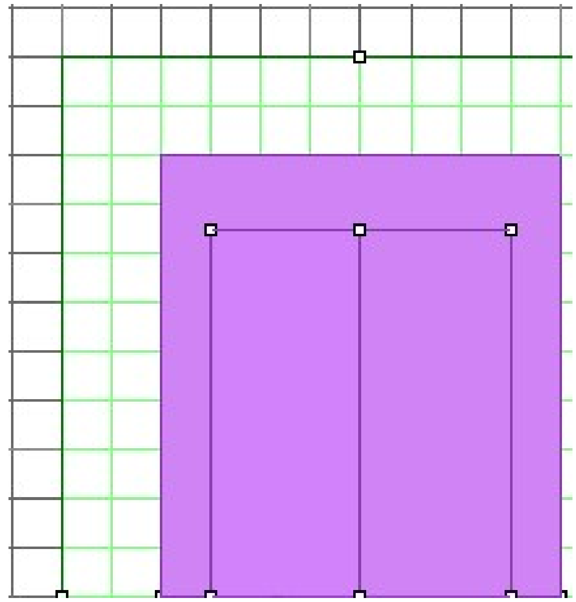
AREAS	POINTS							
101	1	40	2	3	4	5	6	1
102	1	6	7	8	3	2	1	

PLANE RECONSTRUCTION

PLANIMETRIC POINTS			
1	40	2	1

SURFACES	TYPE	NO. OF AREAS	AREAS
1	1 (countryside)	2	101 102

Plane reconstruction



3D BODIES	FACES	POINTS										
1000	1001	3	4	5	6	7	8	3				
	1002	4	11	12	5	4						
	1003	5	6	7	15	14	13	12	5			
	1004	4	3	8	16	10	9	11	4			
	1005	7	8	16	15	7						
	1006	17	19	20	21	17						
	1007	23	24	25	26	23						
	1008	13	14	22	21	13						
	1009	17	18	26	23	22	21	17				
	1010	9	10	18	17	9						
	1014	9	11	12	13	21	20	19	17	9		
	1012	14	15	16	10	18	26	25	24	23	22	14
2000	2001	17	19	20	21	17						
	2002	19	28	29	20	19						
	2003	20	21	22	30	29	20					
	2004	17	18	27	28	19	17					
	2005	28	31	29	28							
	2006	29	30	32	31	29						
	2007	27	28	31	32	27						
	2008	27	32	30	27							
	2009	27	30	22	23	36	33	34	39	26	18	27
	2010	33	36	35	39	34	33					
	2011	35	36	37	35							
	2012	35	37	38	35							
	2013	35	38	39	35							
	2014	23	24	37	36	23						
	2015	24	25	38	37	24						
	2016	25	26	39	38	25						
	2017	23	24	25	26	23						
	2019	17	18	26	23	22	21	17				

SOLID RECONSTRUCTION

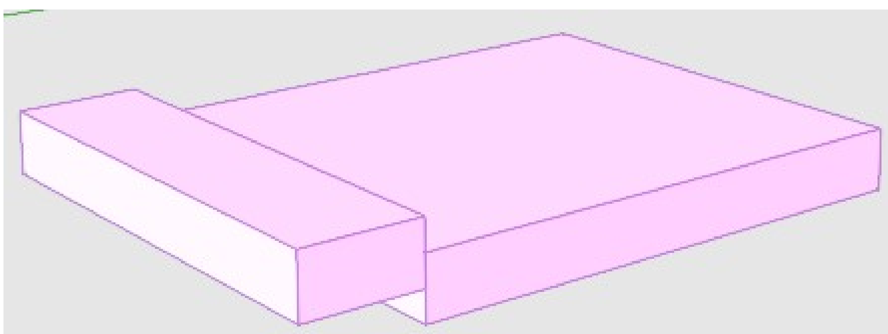
3D BODIES	NO. OF BODIES NO. OF REGIONS	3D BODIES AND THEIR REGIONS REGIONS AND THEIR 3D BODIES		
1000	3 3	2000 2001	2000 2017	2000 2019
2000	3 3	1000 1006	1000 1007	1000 1009

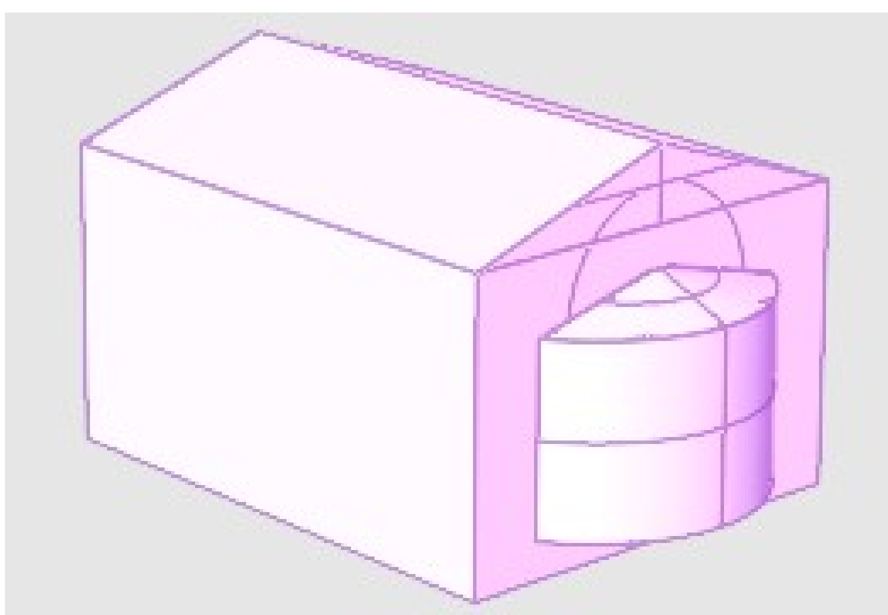
BUILDING	3D BODIES	NO. OF POINTS	POINTS						
1	1	7	1 3	2 4	3 5	4 6	5 7	6 8	1 3

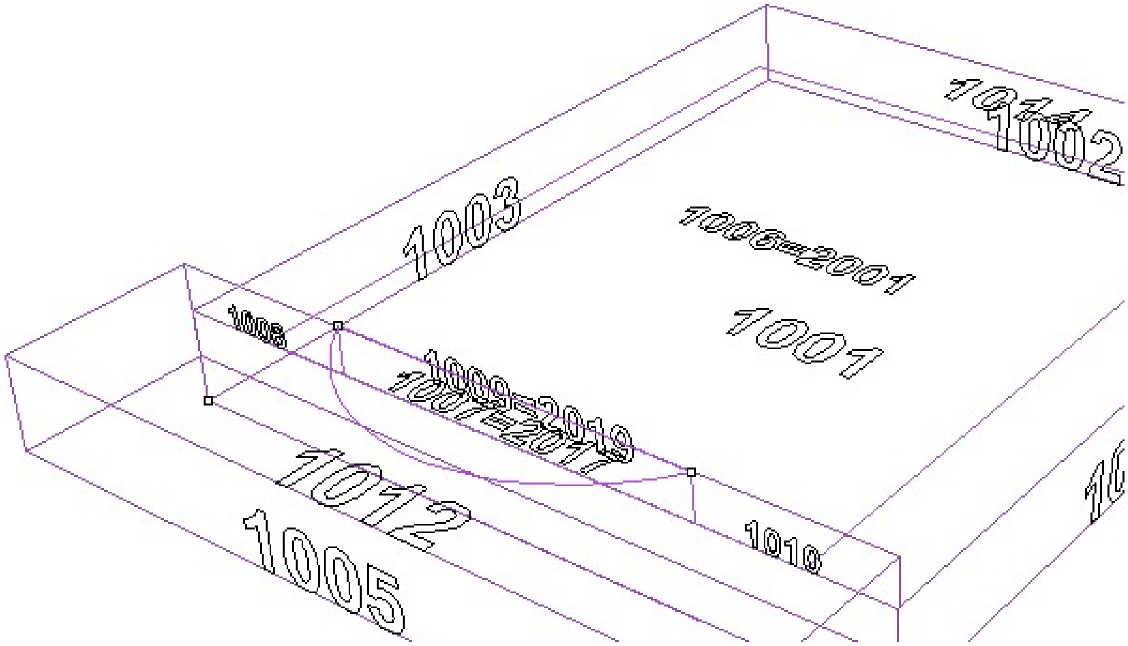
3D BODIES	NO. OF FACES	FACES
1	23	1002 1003 1004 1005 1008 1010 1014 1012 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016

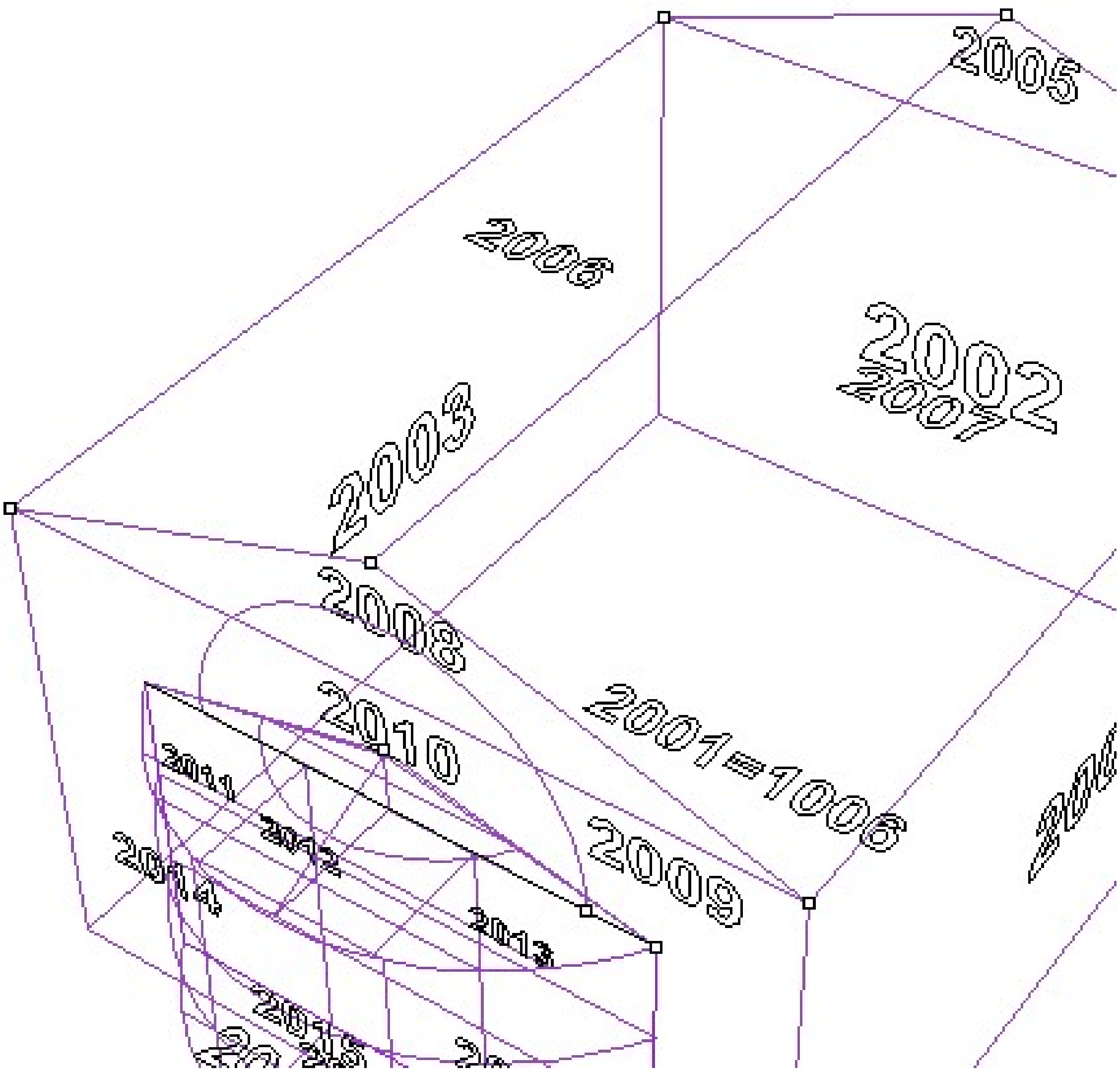
3D BODIES	NO. OF FACES	FACES
1	1	1001

AXONOMETRY



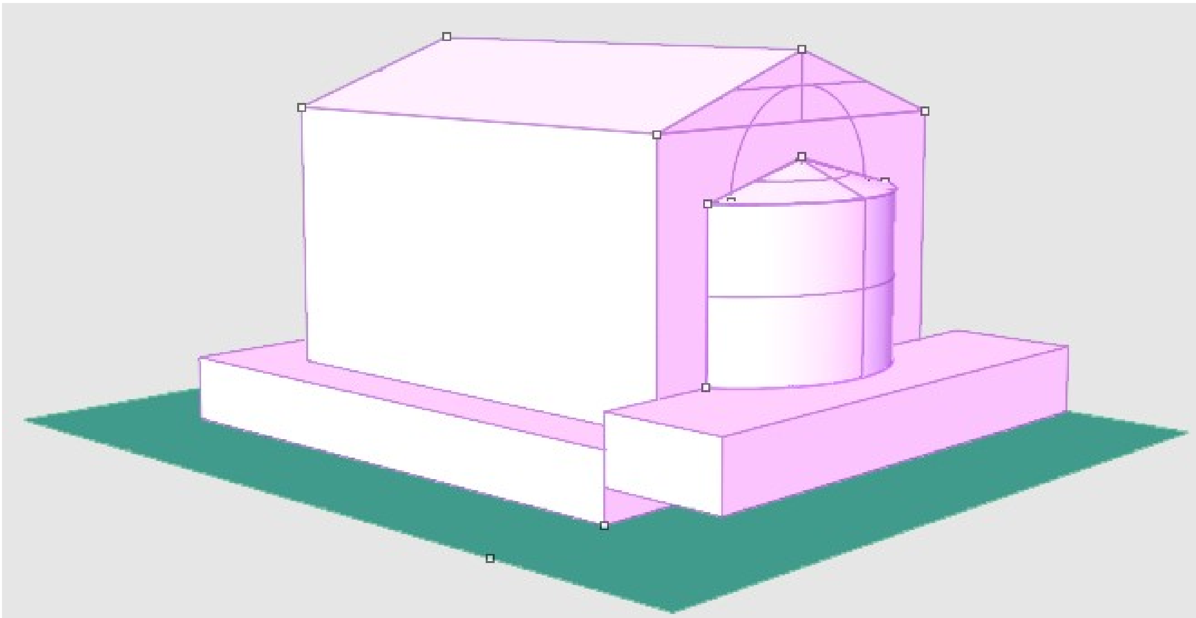



1014
1002
1003
1006=2001
1001
1008
1009=2019
1007=2017
1012
1005
1010

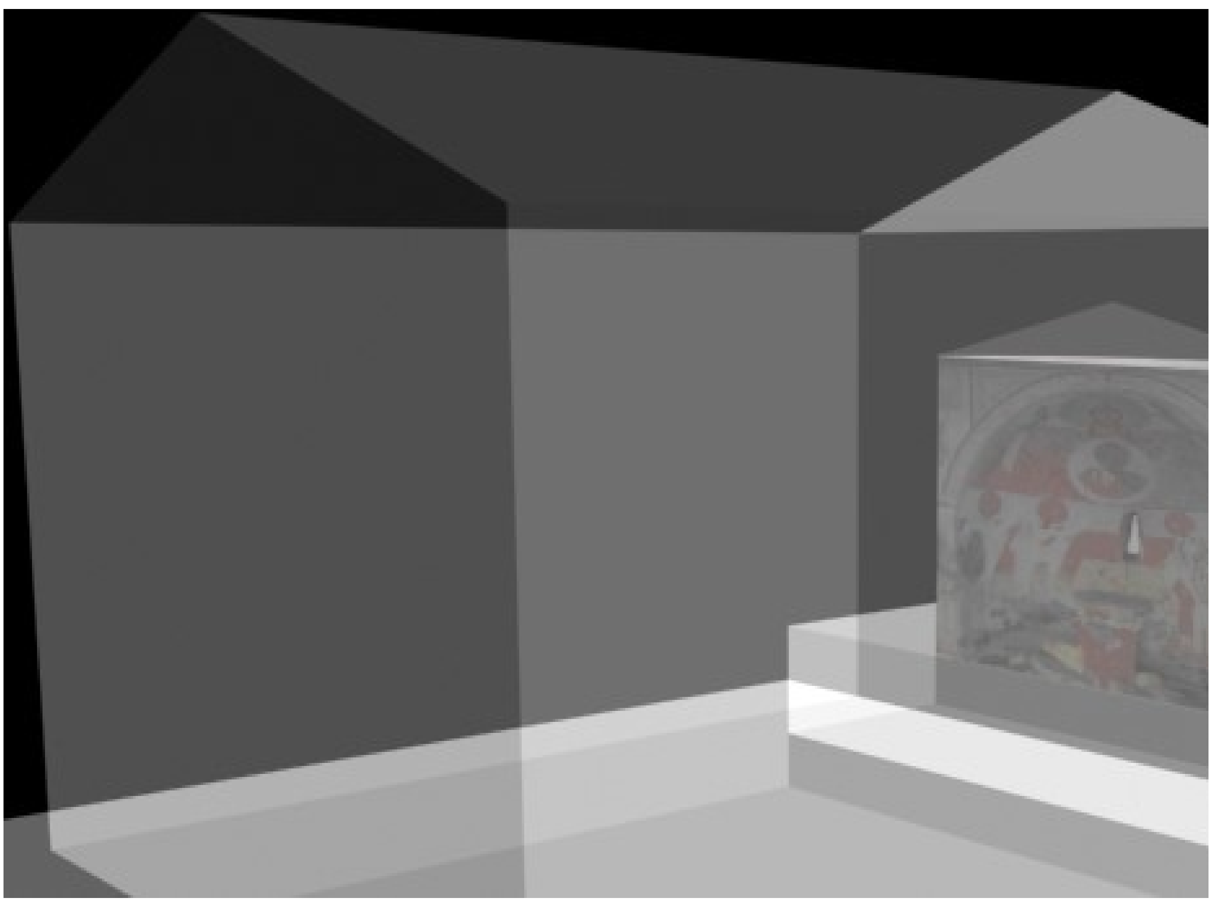

2005
2006
2002
2007
2003
2008
2010
2001=1006
2011
2012
2014
2009
2013

FINAL RECONSTRUCTION

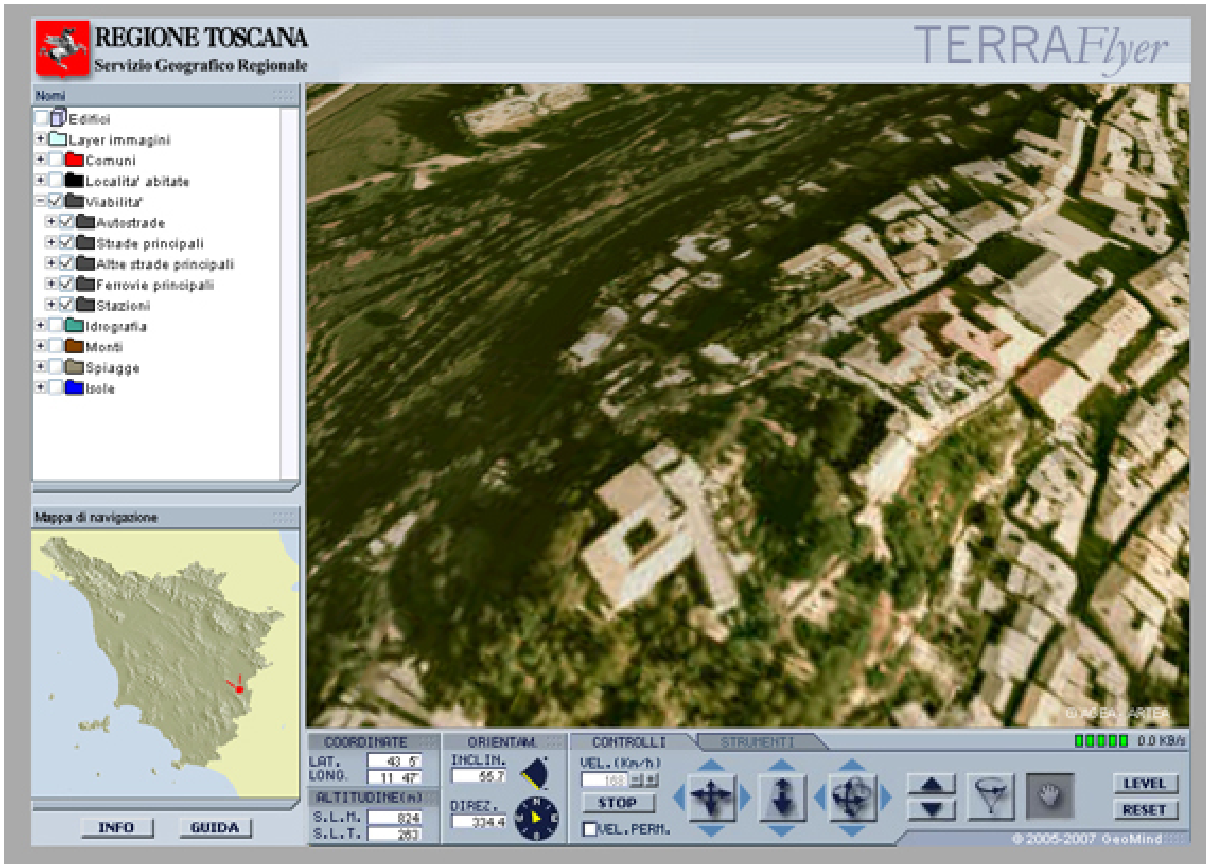
TOPOLOGY



THEMATISM

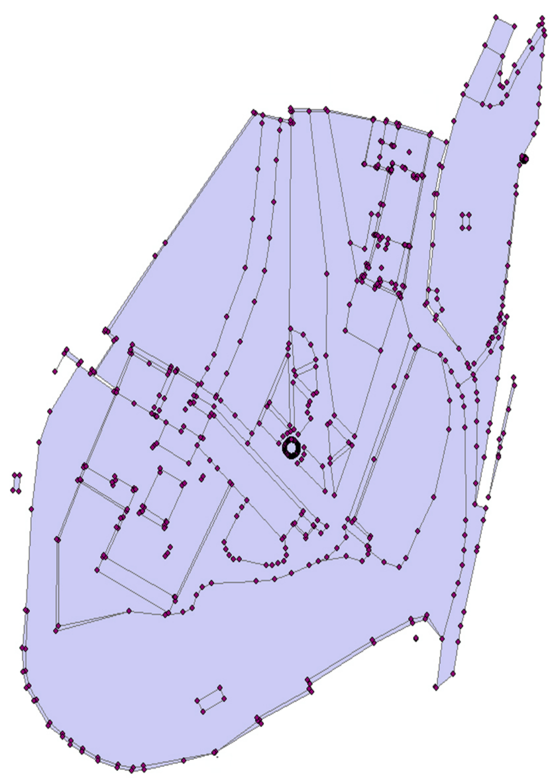


AGAINST WAR AND HUMAN MADNESS !!



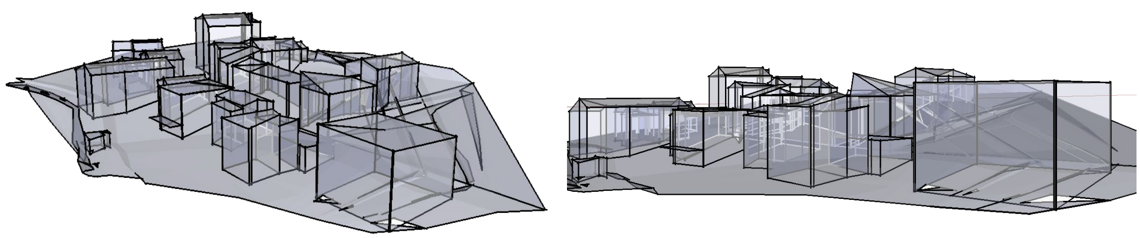
Montepulciano (Siena) in Tuscany

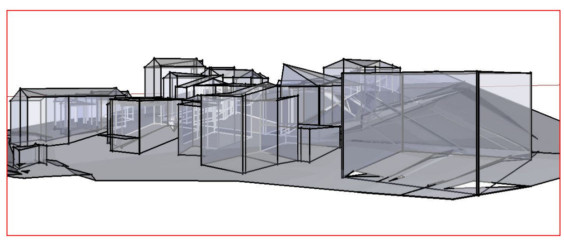
Points in the area



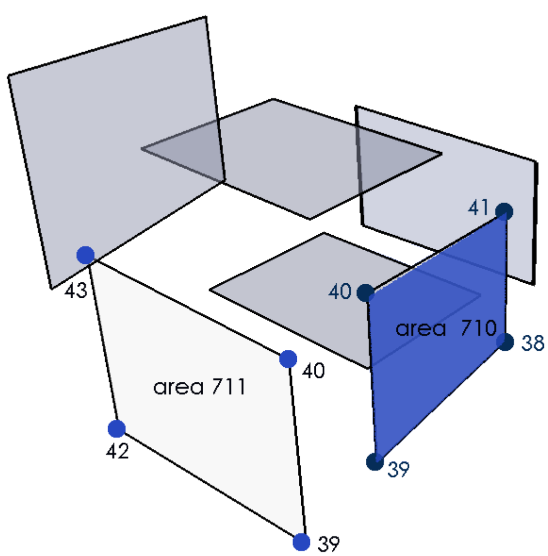
Regions in the area

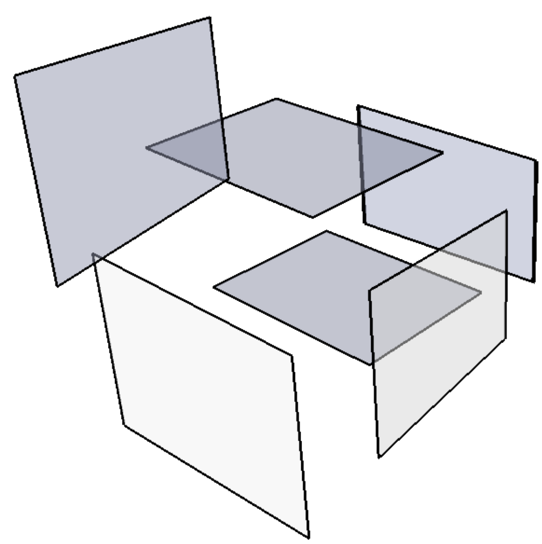
Lines and 3D bodies





Reduction to sides and regions
and uncomposing 3D bodies

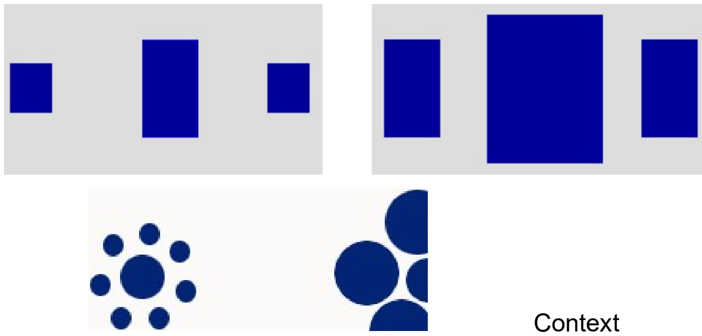




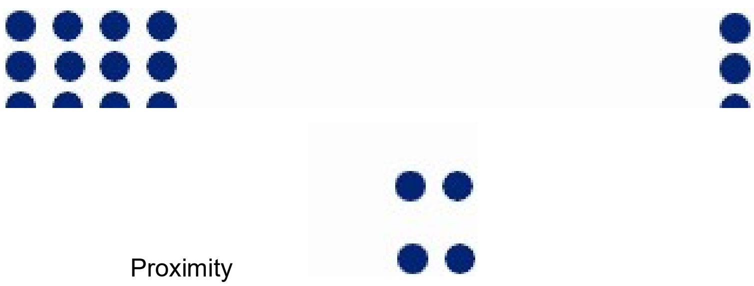
Appendix A – Gestalt theory and perceptual grouping

Principles of Gestalt theory

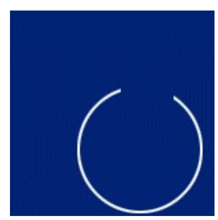
The Gestalt principles are a set of laws describing how typically to see objects by grouping similar elements, recognizing patterns, simplifying complex images and using the standard “tricks” of perspective design.



Context

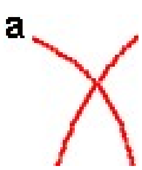


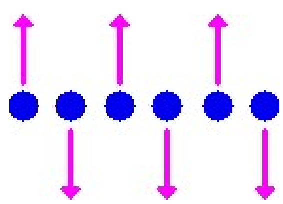
Proximity



Closure

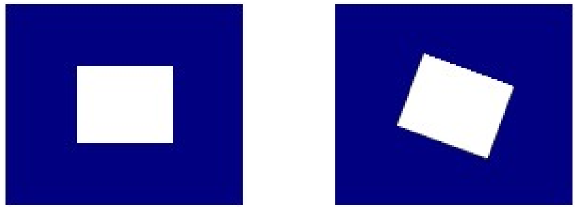
Continuity

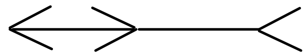


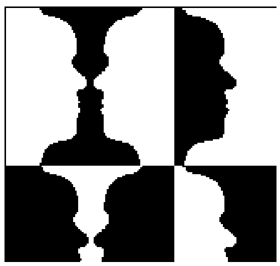


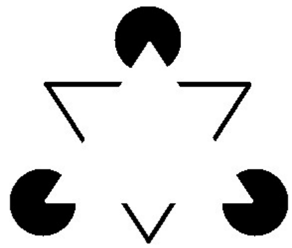
Common movement

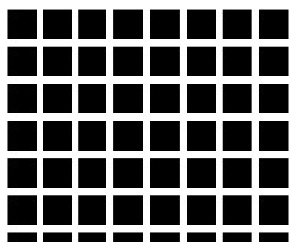
Harmony







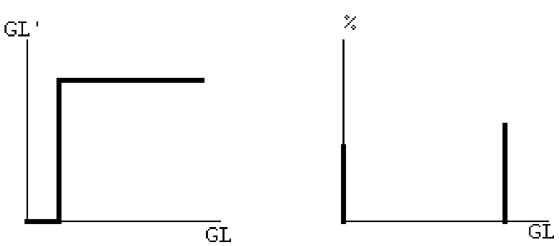




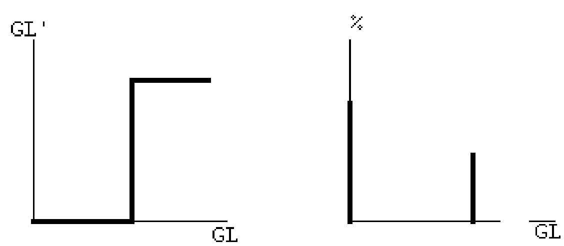
Minimum amount of information (looking for figure and background)

Discrete versus continuous processing

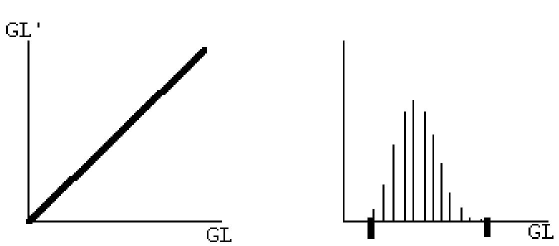
Low binarization threshold



High binarization threshold

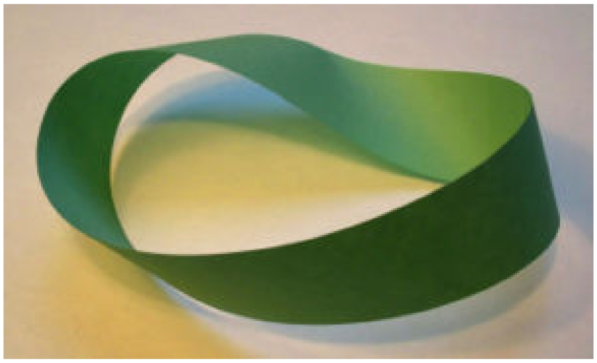


Linear and normal distributed data



Data's equalization

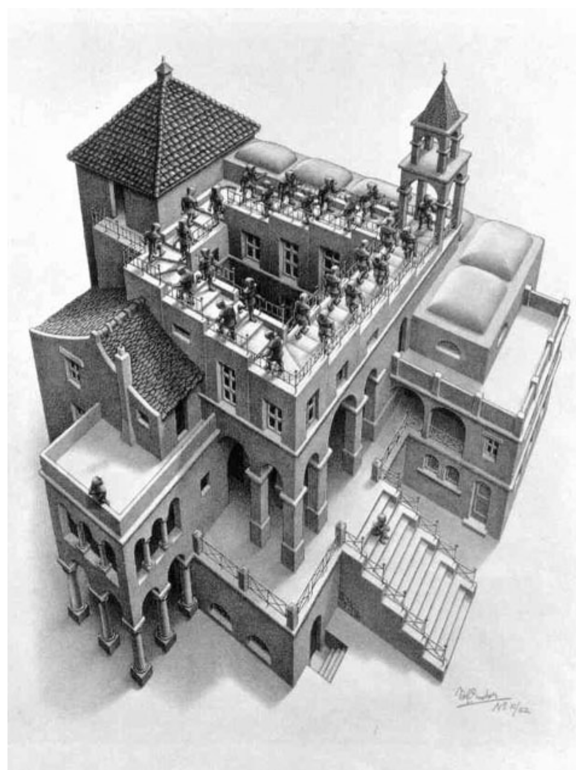
Hambiguity drabacks



Moebius' ring

Klein's bottle

Maurits Cornelis Escher, Water fall, 1961

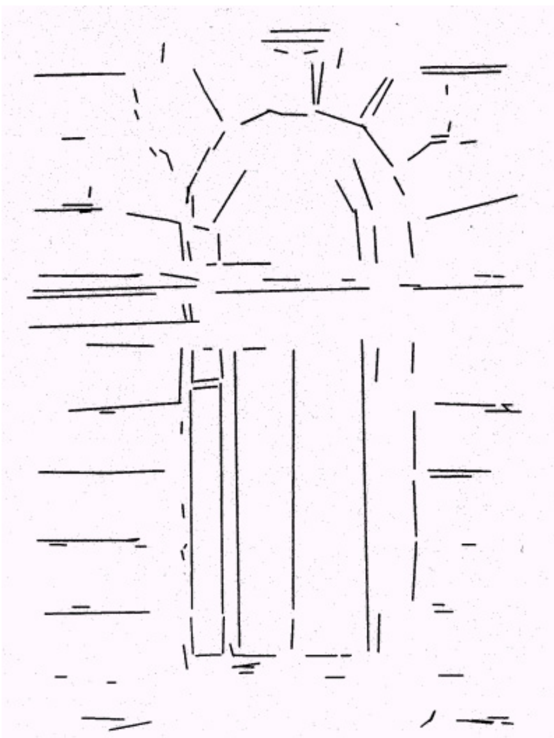


Ascent descent, 1960

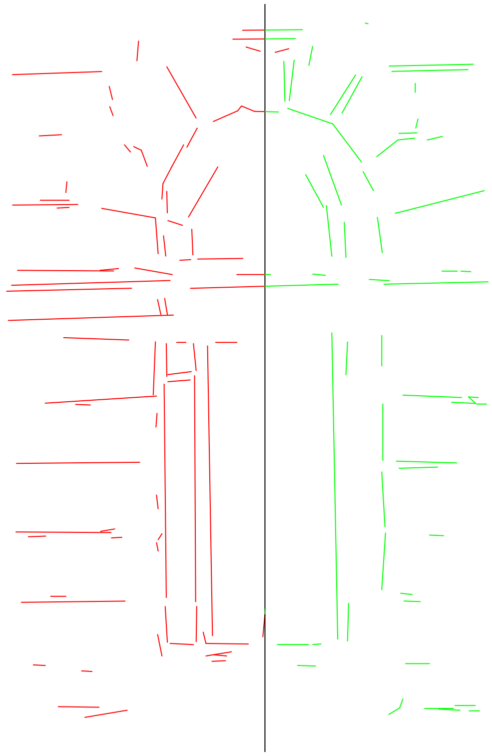
Ambiguity is often associated with negative responses and ambiguity seems restricted to a few situations, such as art. Nevertheless theories of judgment formation, especially the processing fluency account, suggest that non-ambiguous stimuli are processed faster and therefore are preferred to ambiguous stimuli, which are hard to process.

Perceptual grouping

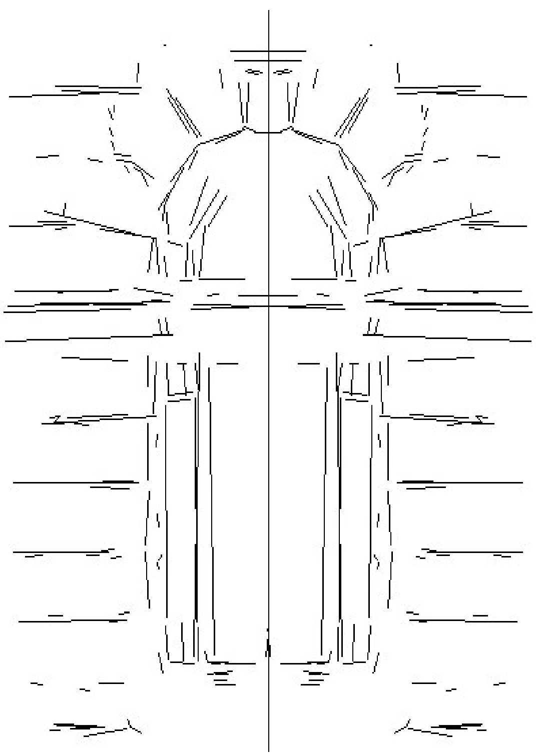
Perceptual system processes global features of objects to identify them and proceeding to a finer analysis.



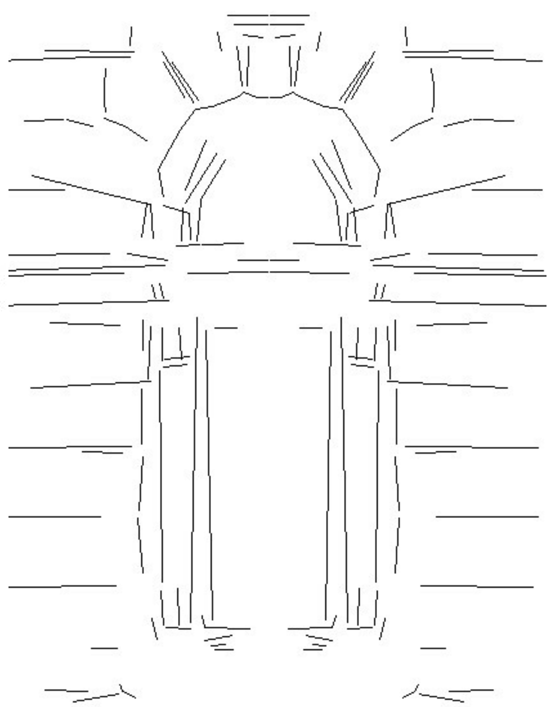
Grouping of 126 data among 500 ones



Replacing the data by symmetry

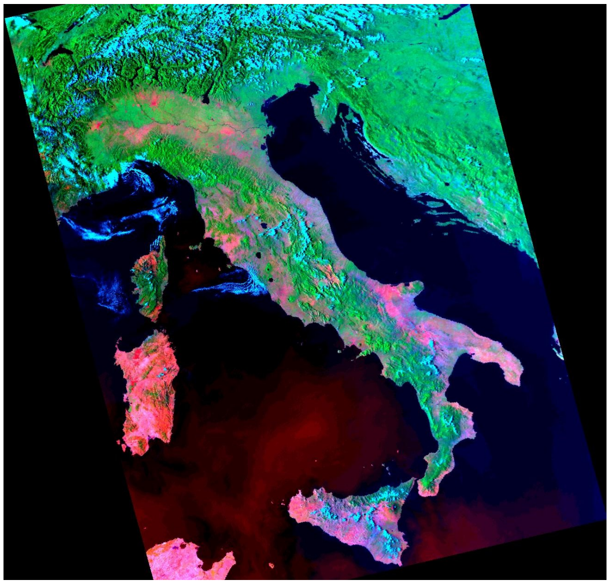


Copy the left part in the right one and vice-versa

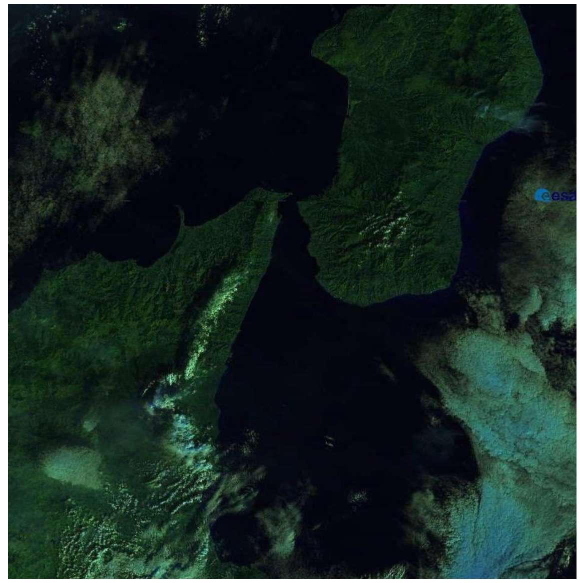


Erasing insignificant data

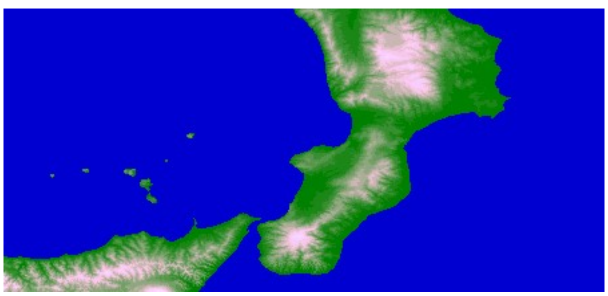
Appendix B – Regional and local GIS



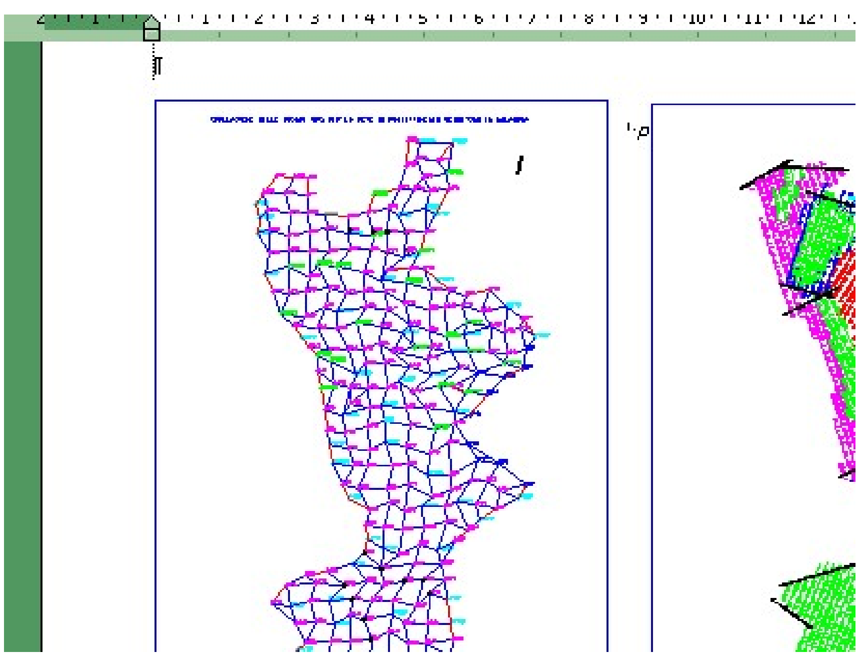
Italy (NOAA image)



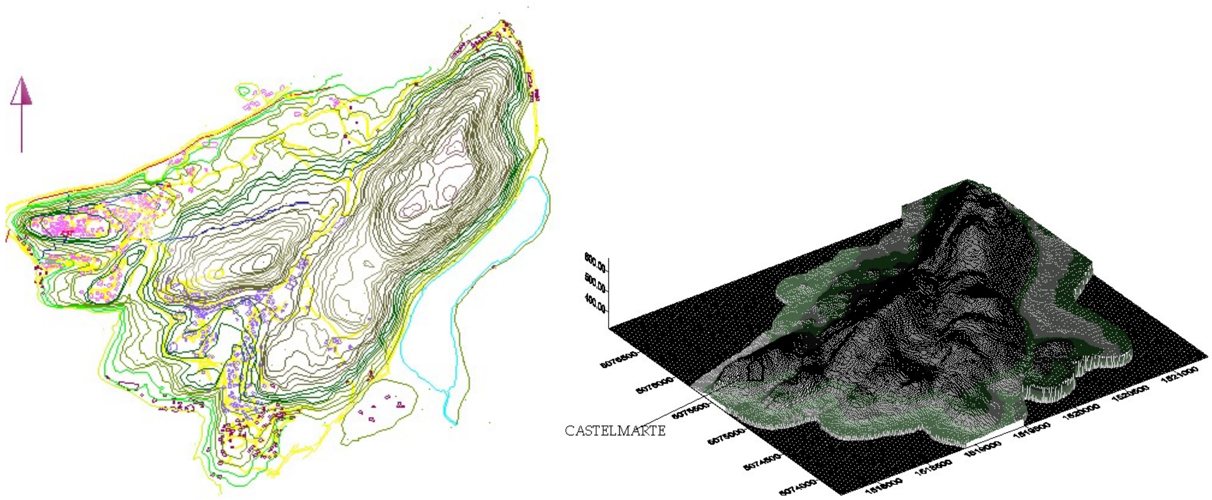
Calabria Region (ERS1 immagine radar)



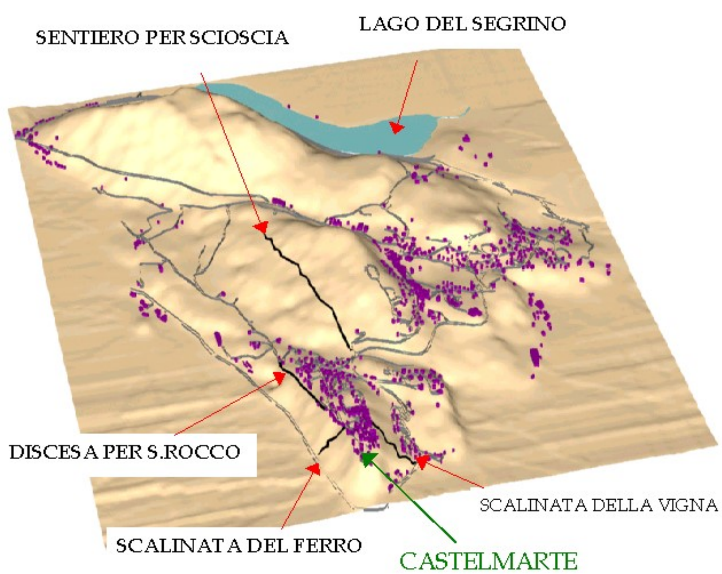
Shading and 2D ½ DSM of Calbria Region and East Sicily



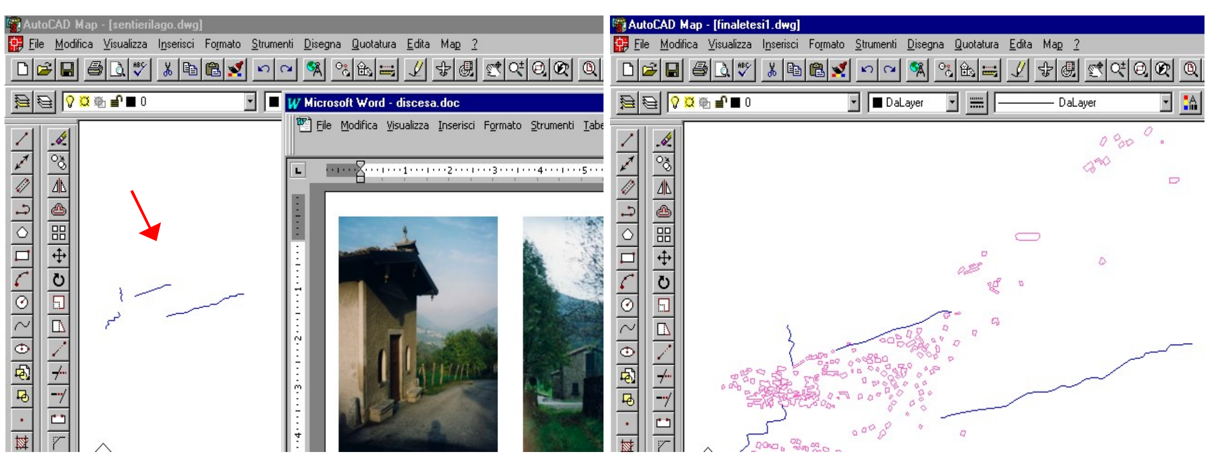
Geodetic network and imagery coverage of Calabria region



Vector map in Brianza   2D ½ DSM

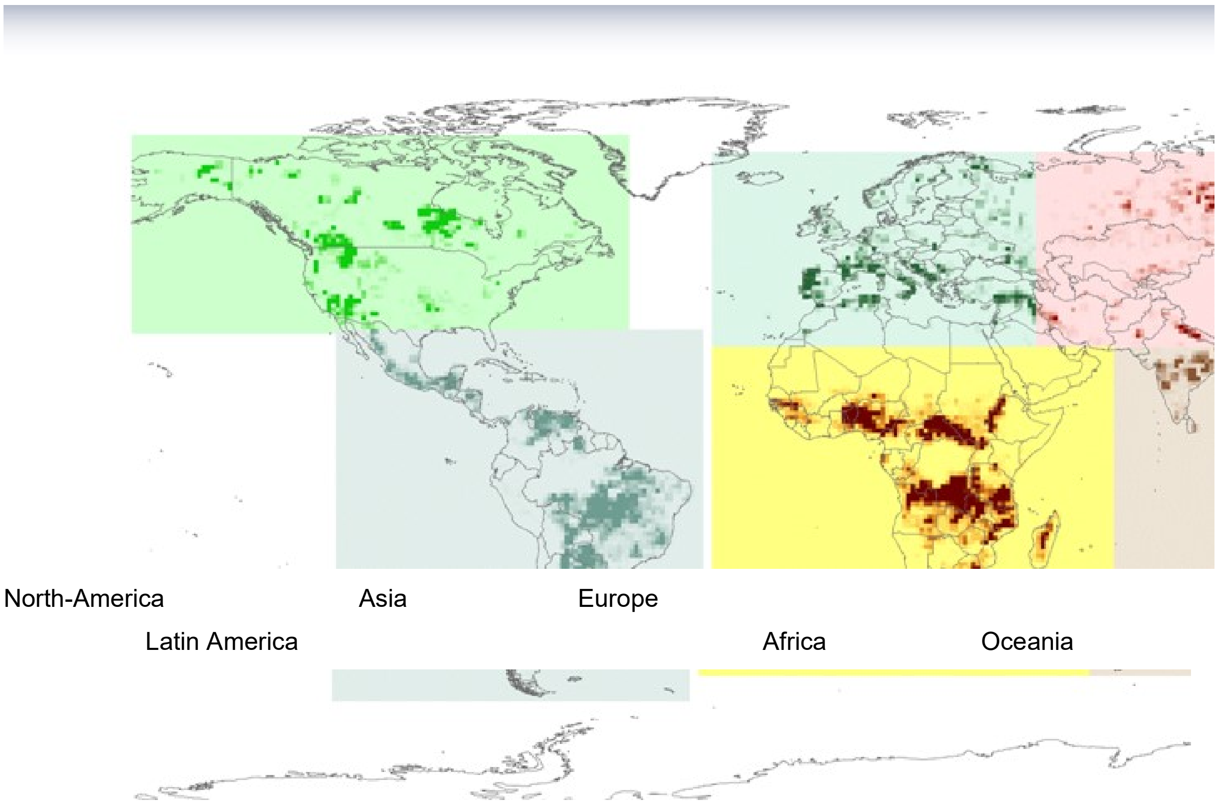


2D ½ DSM and Texture mapping

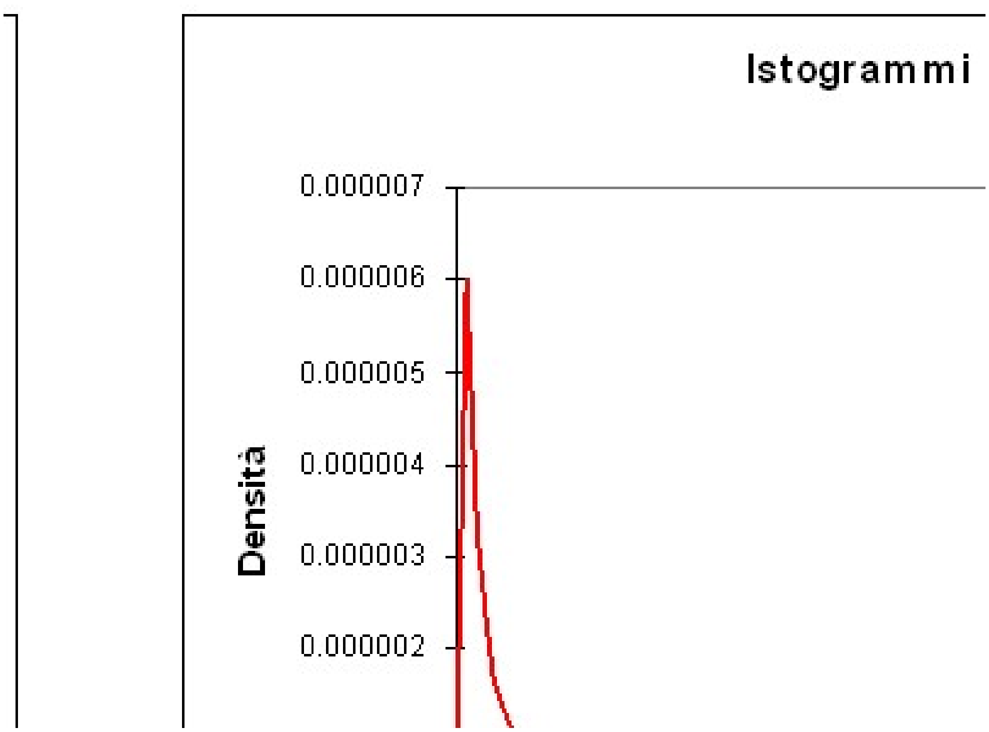


Documents and queries

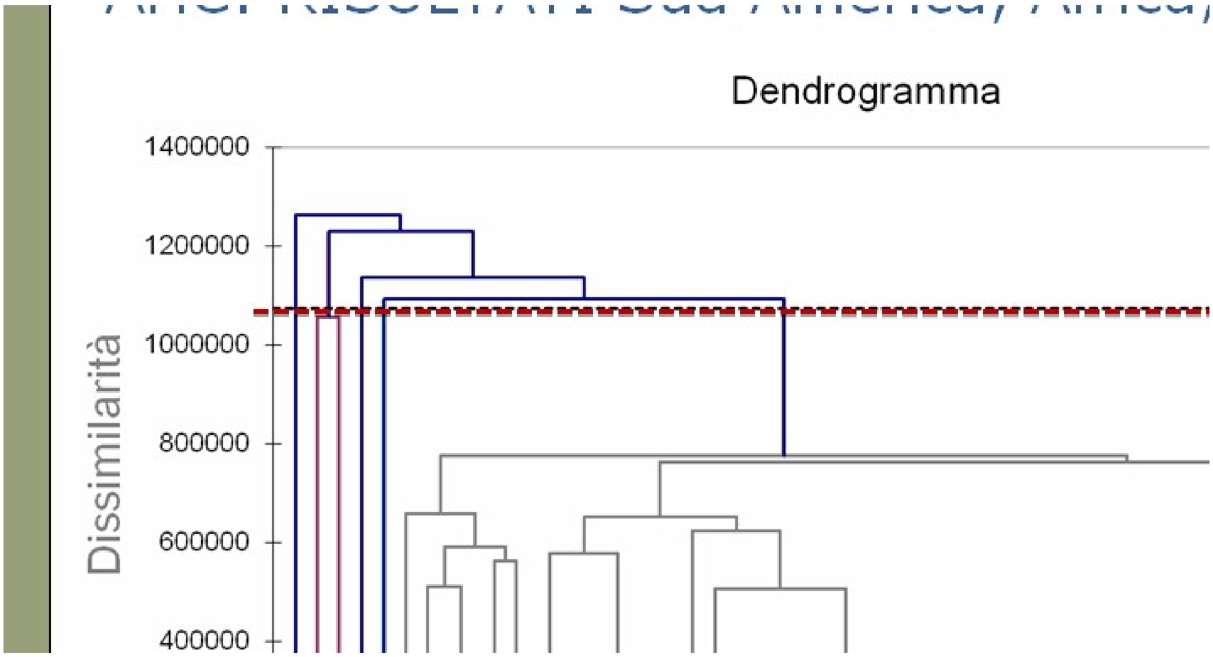
Appendix C – Remote sensing and Geomatics: two examples



CO emissions

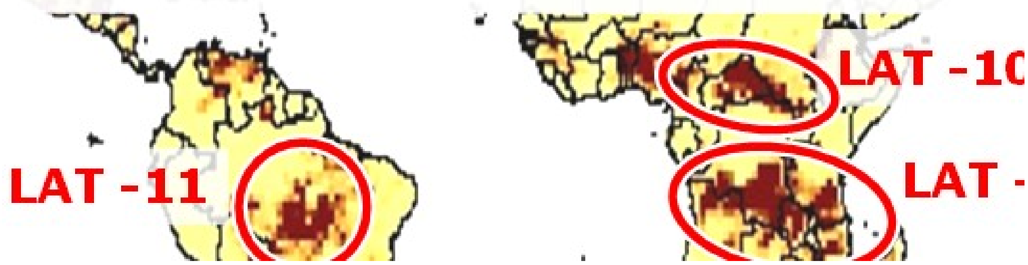


Histogram and probability of CO emissions in Africa

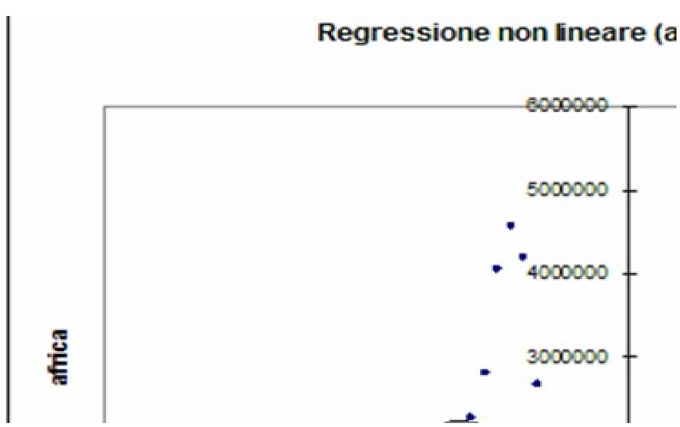


Cluster analysis of CO emissions in center Africa

baric	157660.939	528536.626	0.000
baric	1185255.264	528536.626	0.000
baric	4350027.887	528536.626	0.000

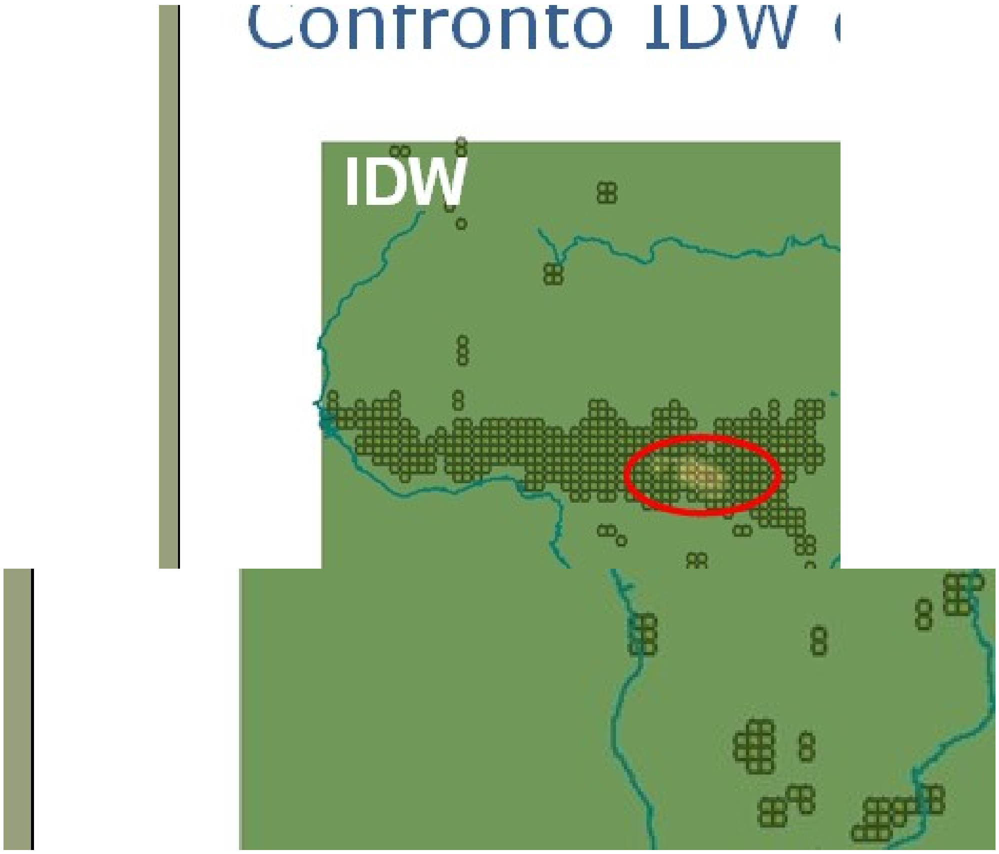


Cluster analysis of CO emissions in Latin America, Africa and Oceania

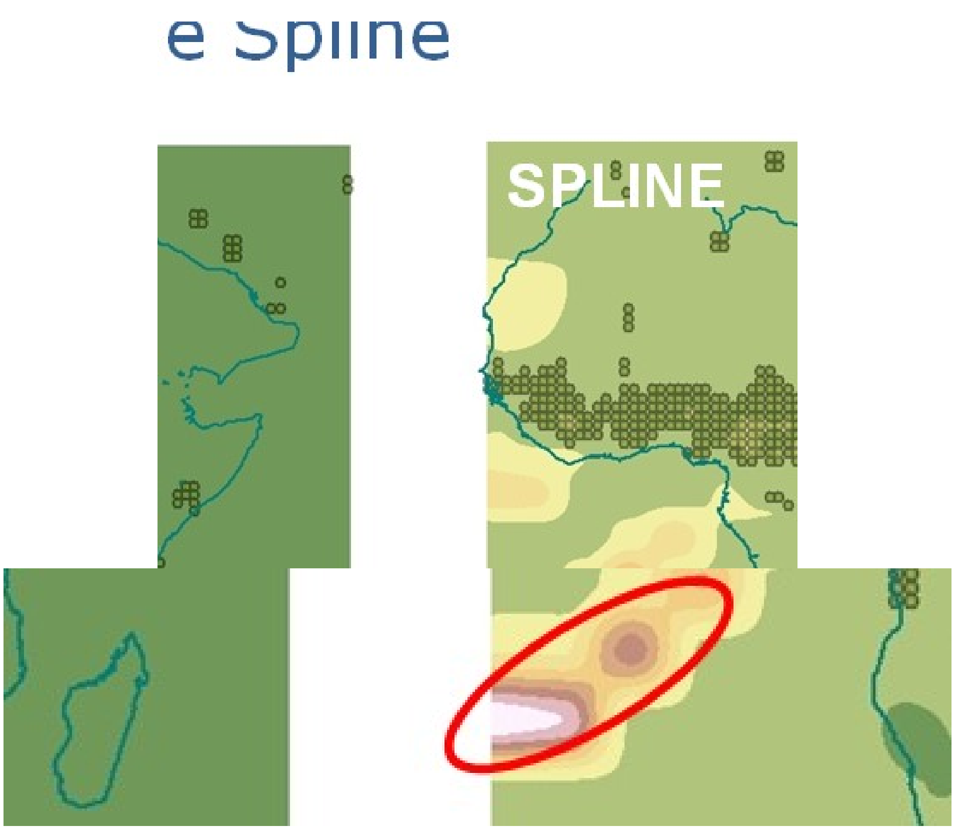


Latitude

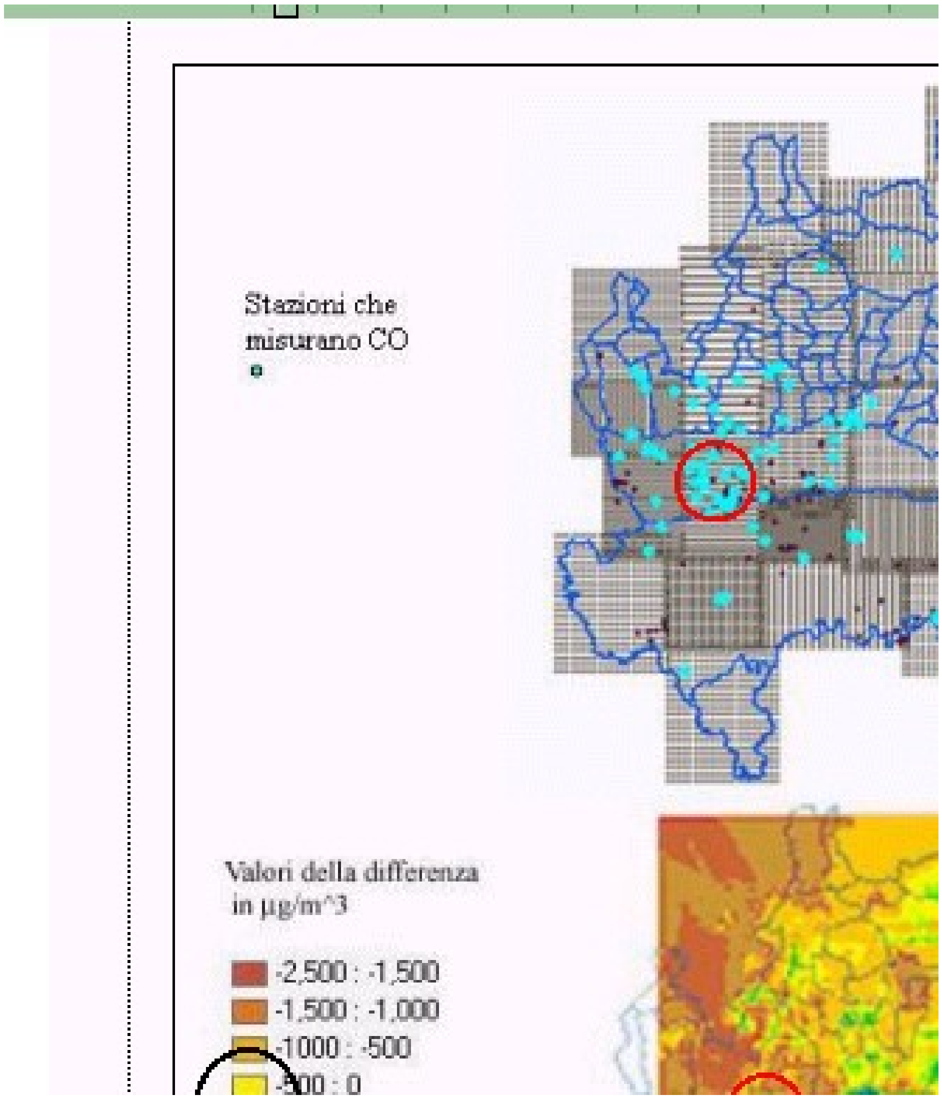
Distribution and non-linear regression of CO emissions in Africa

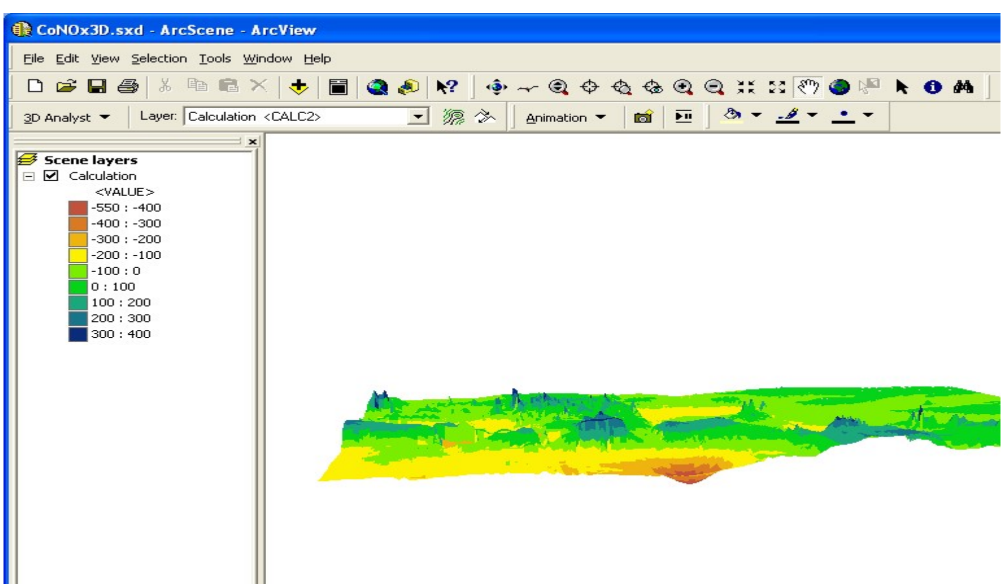


Inverse distance weighting interpolation of CO emissions in Africa

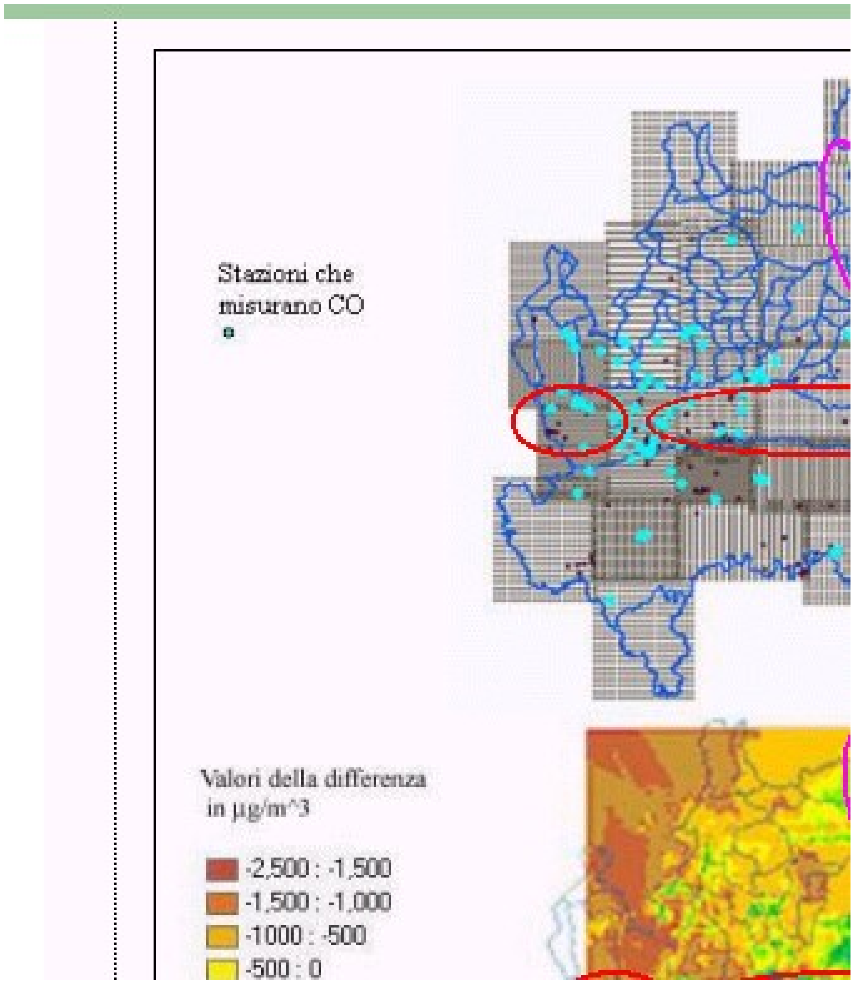


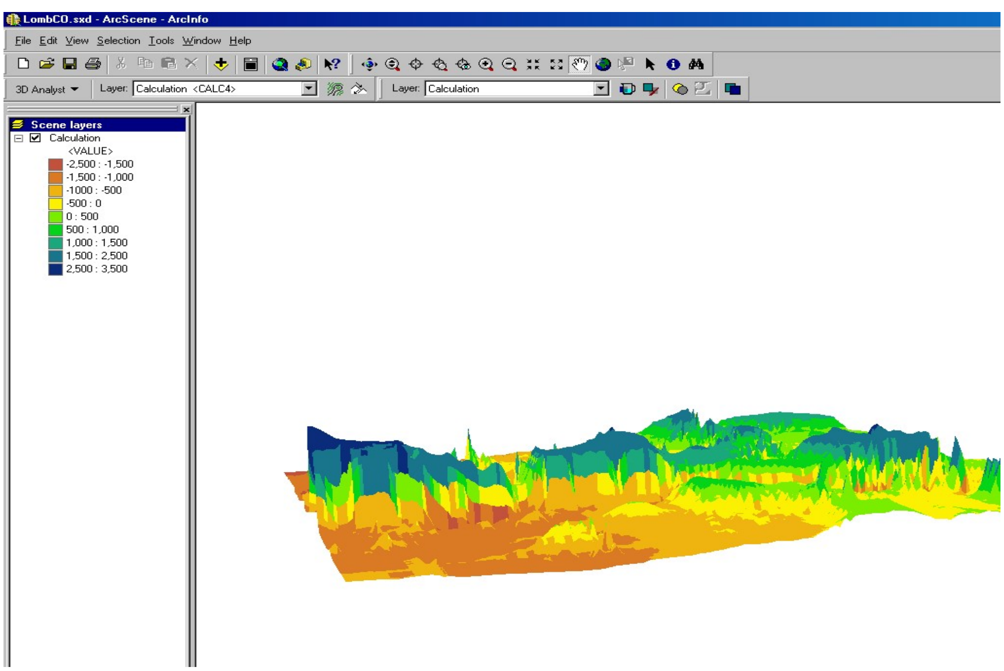
Spline interpolation of CO emissions in Africa



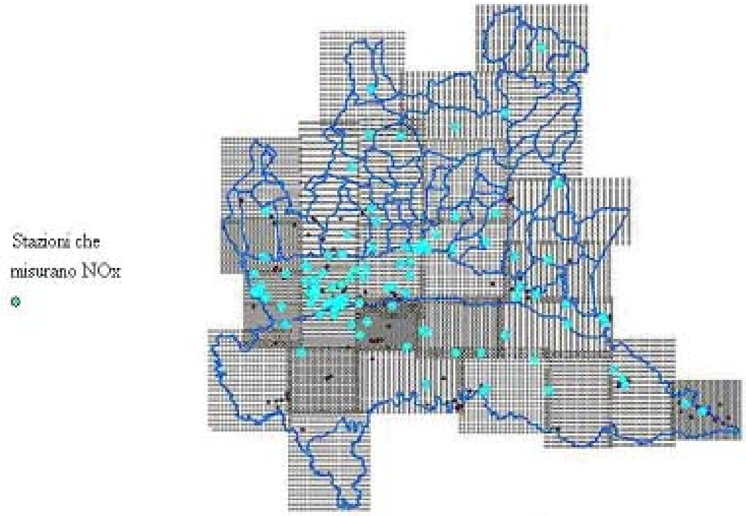


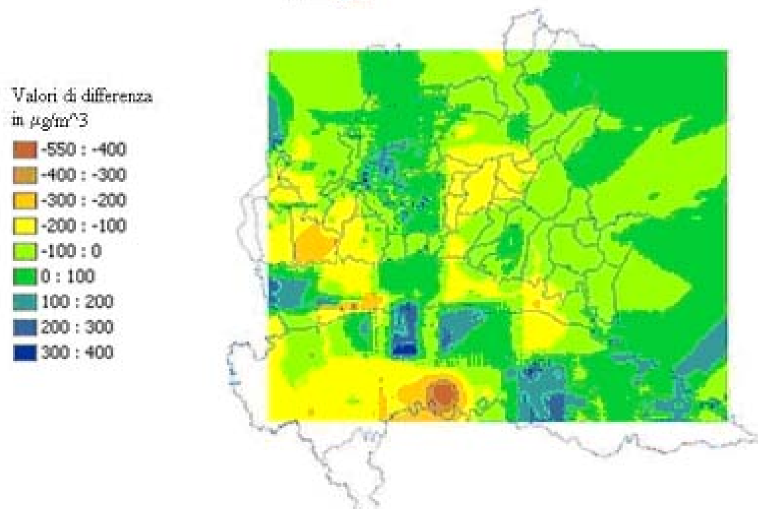
Pollution in Lombardy Region in fall (data acquisition, mapping and 2D ½ DSM)

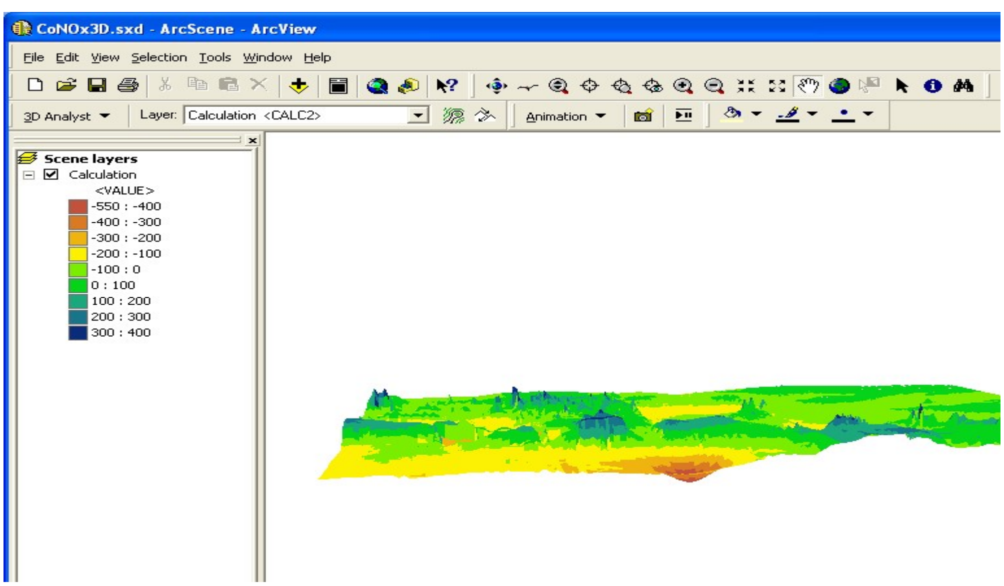




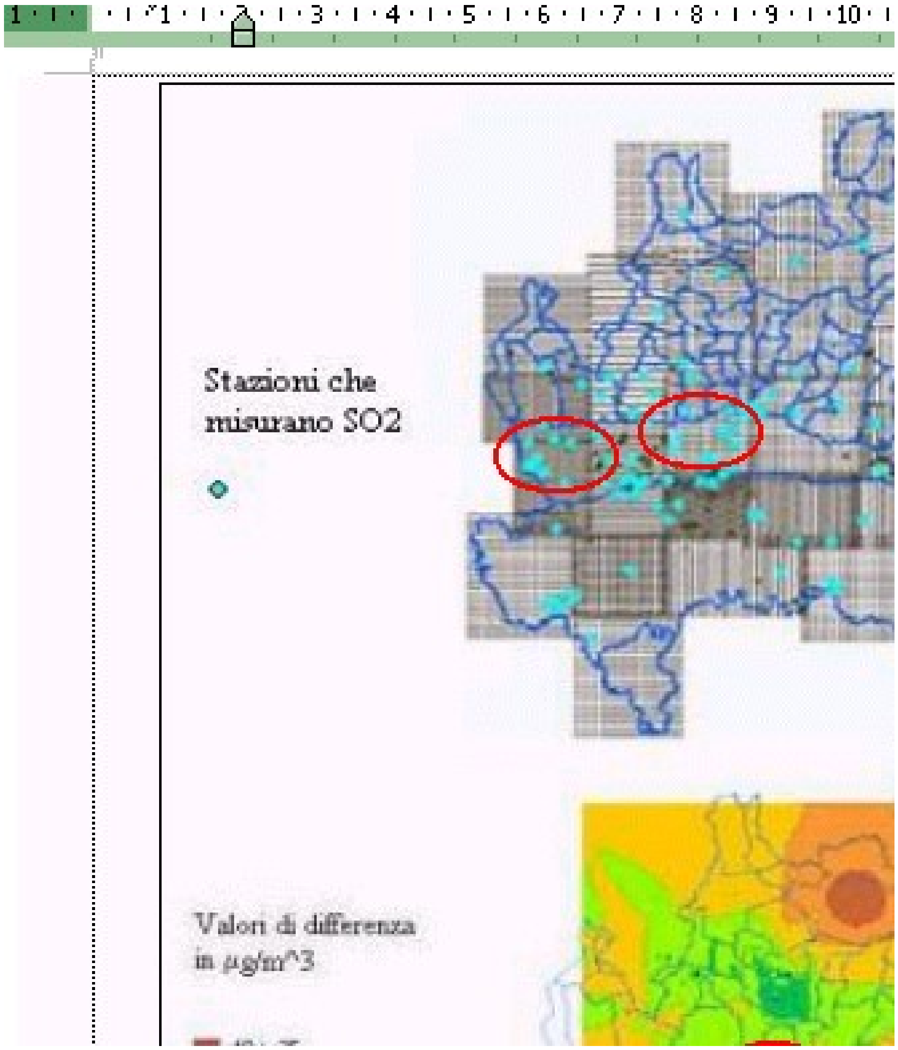
Pollution in Lombardy Region in winter (data acquisition, mapping and 2D ½ DSM)

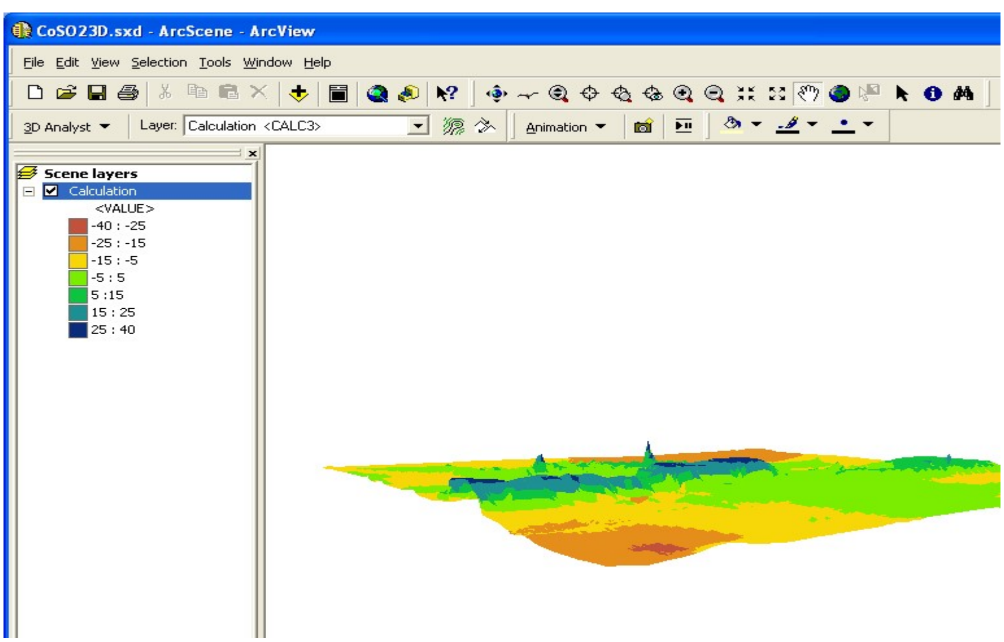






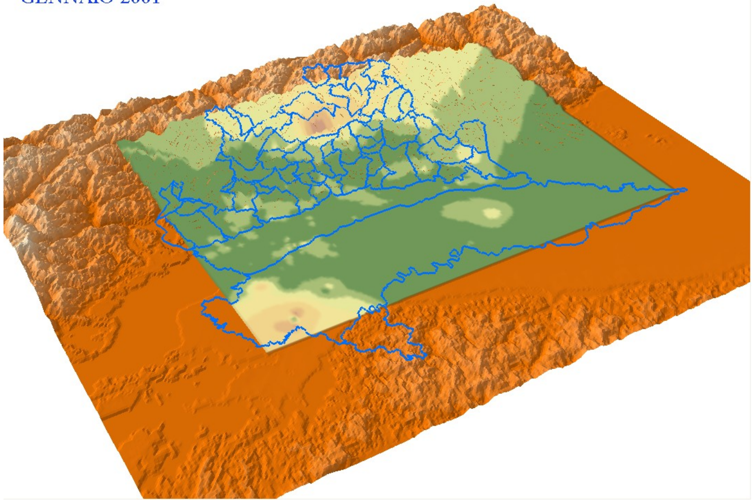
Pollution in Lombardy Region in spring (data acquisition, mapping and 2D ½ DSM)



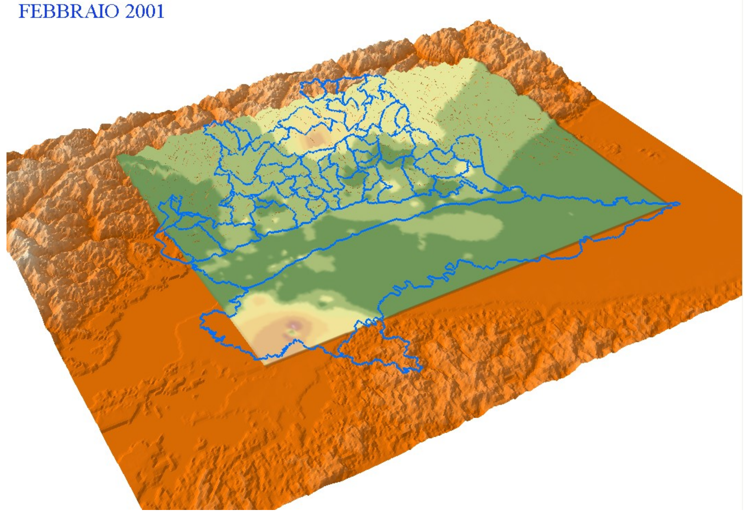


Pollution in Lombardy Region in summer (data acquisition, mapping and 2D ½ DSM)

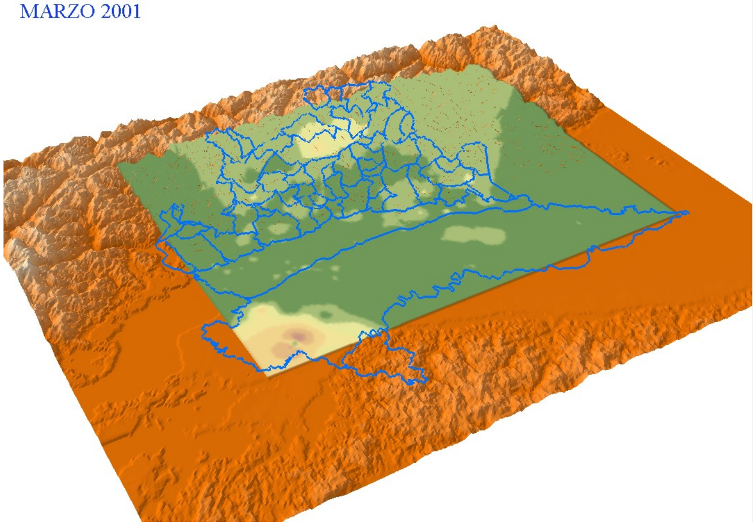
GENNAIO 2001



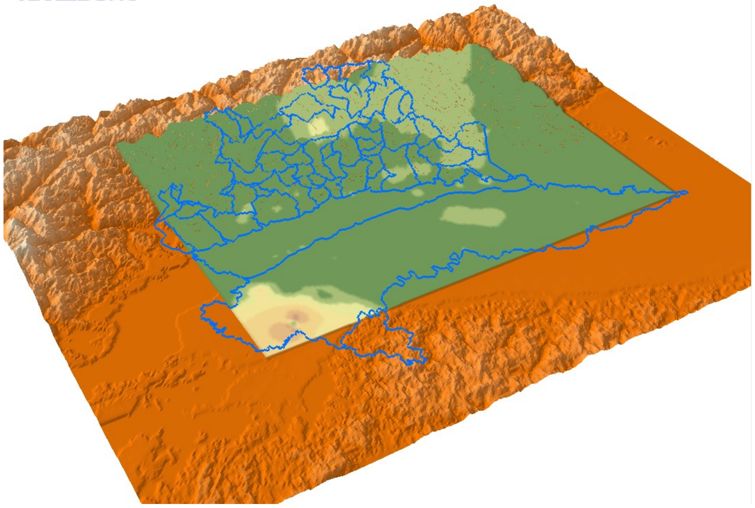
FEBBRAIO 2001



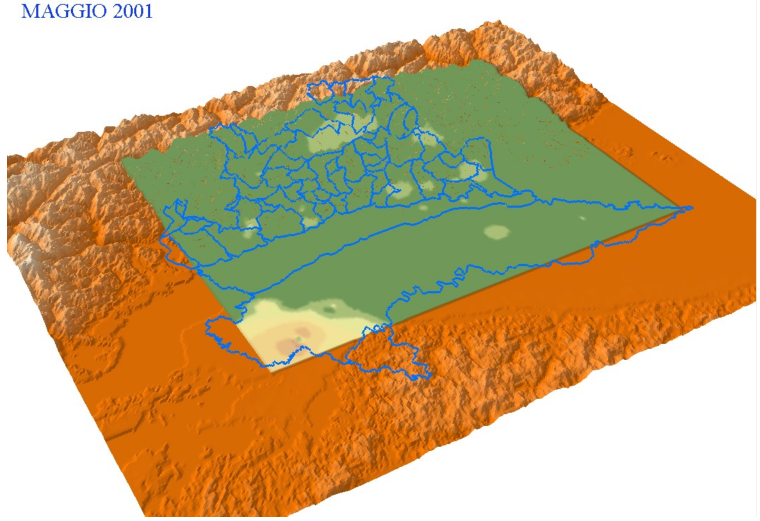
MARZO 2001



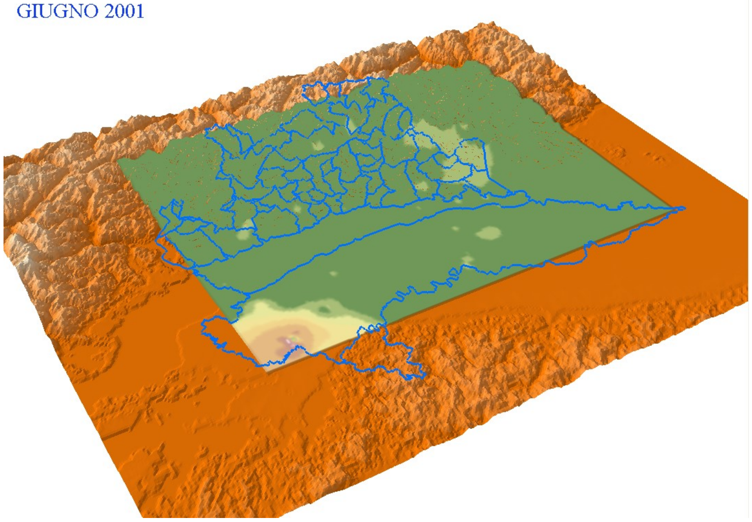
APRILE 2001



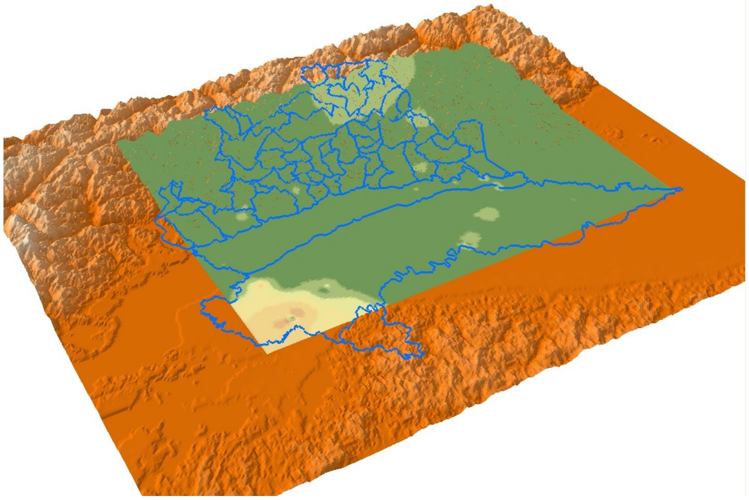
MAGGIO 2001



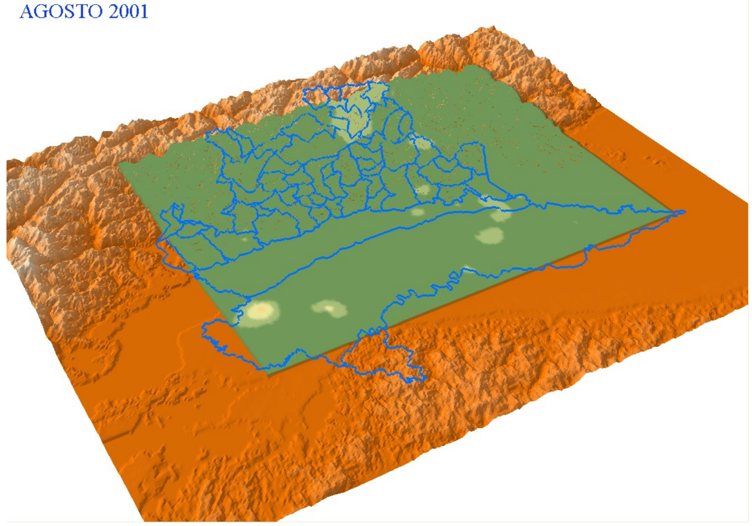
GIUGNO 2001



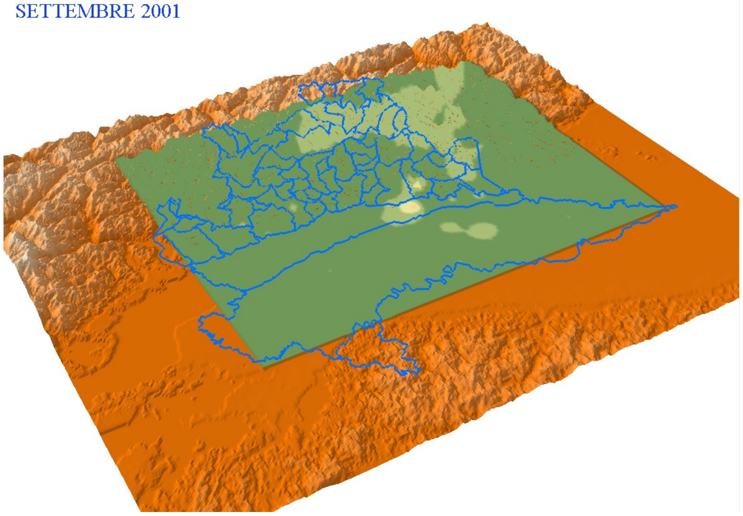
LUGLIO 2001



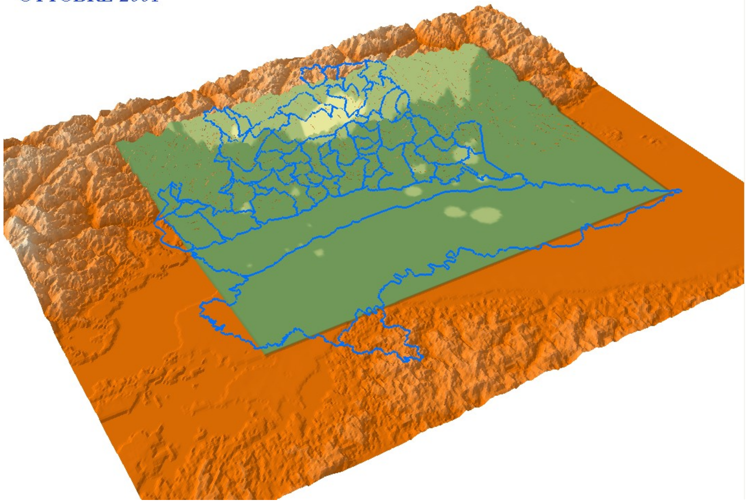
AGOSTO 2001



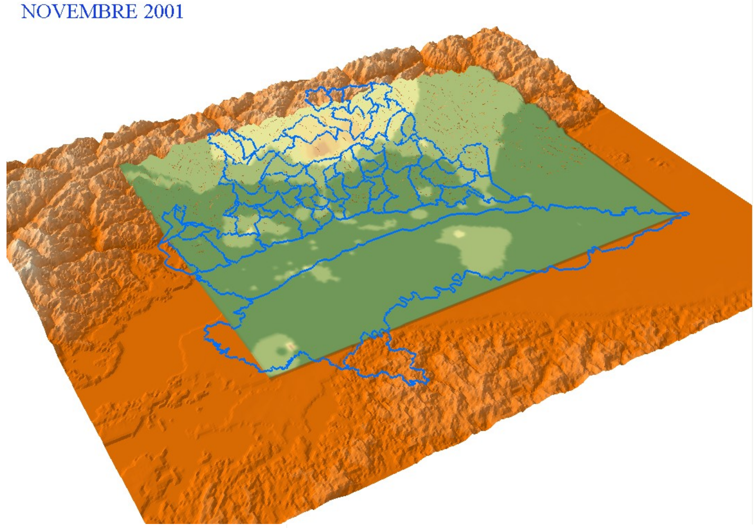
SETTEMBRE 2001



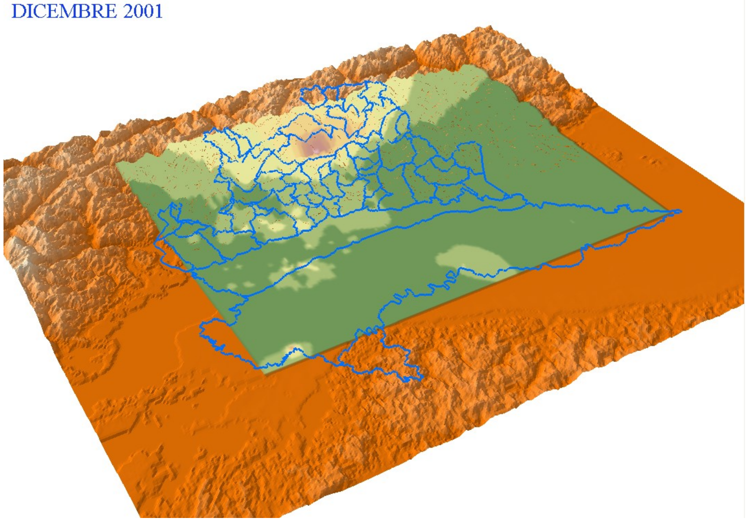
OTTOBRE 2001



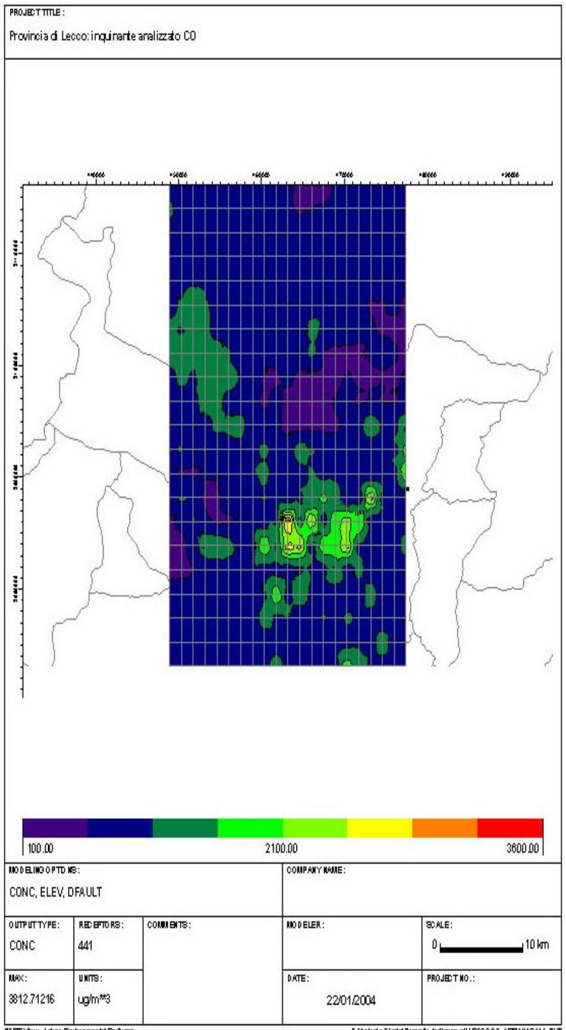
NOVEMBRE 2001

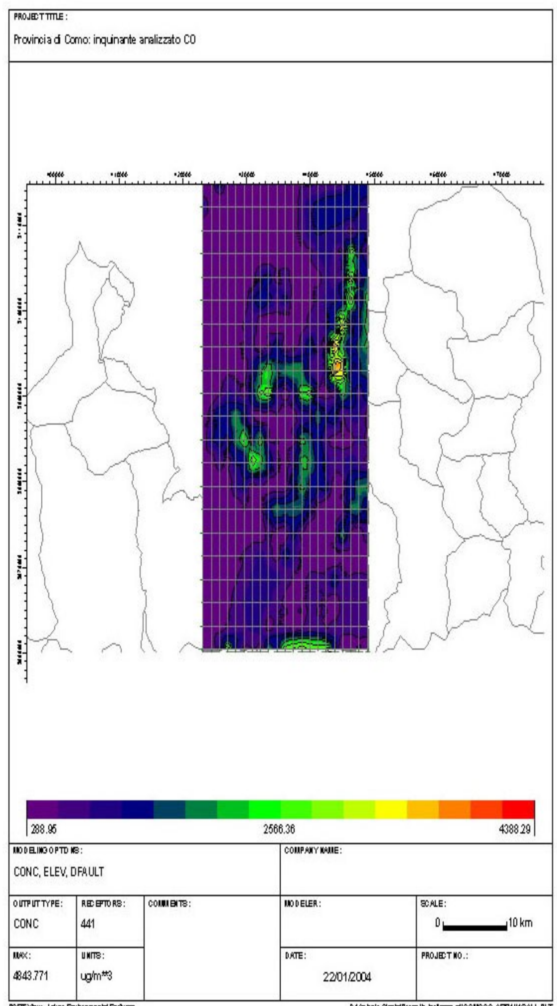


DICEMBRE 2001



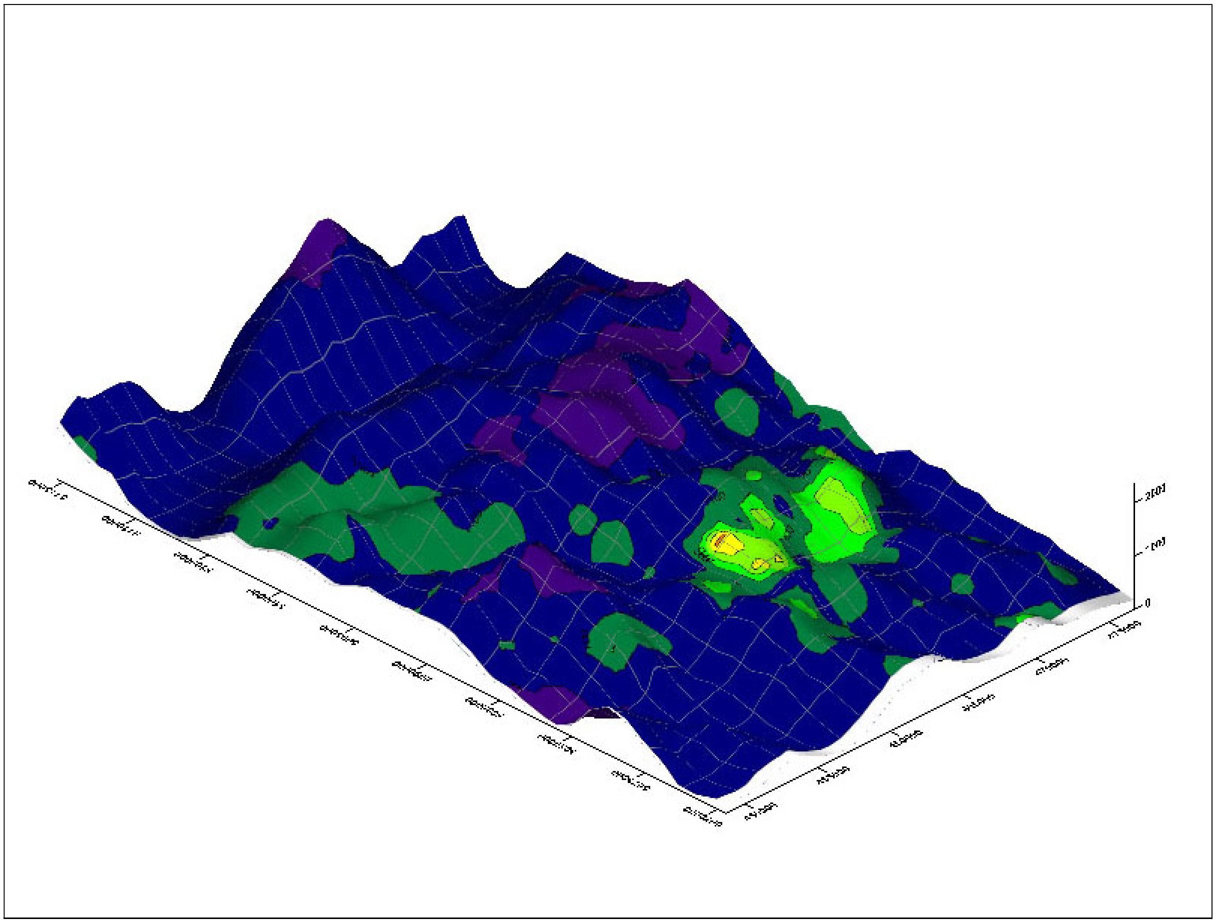
Pollution in Lombardy Region mounth by mounth (2D ½ DSM)



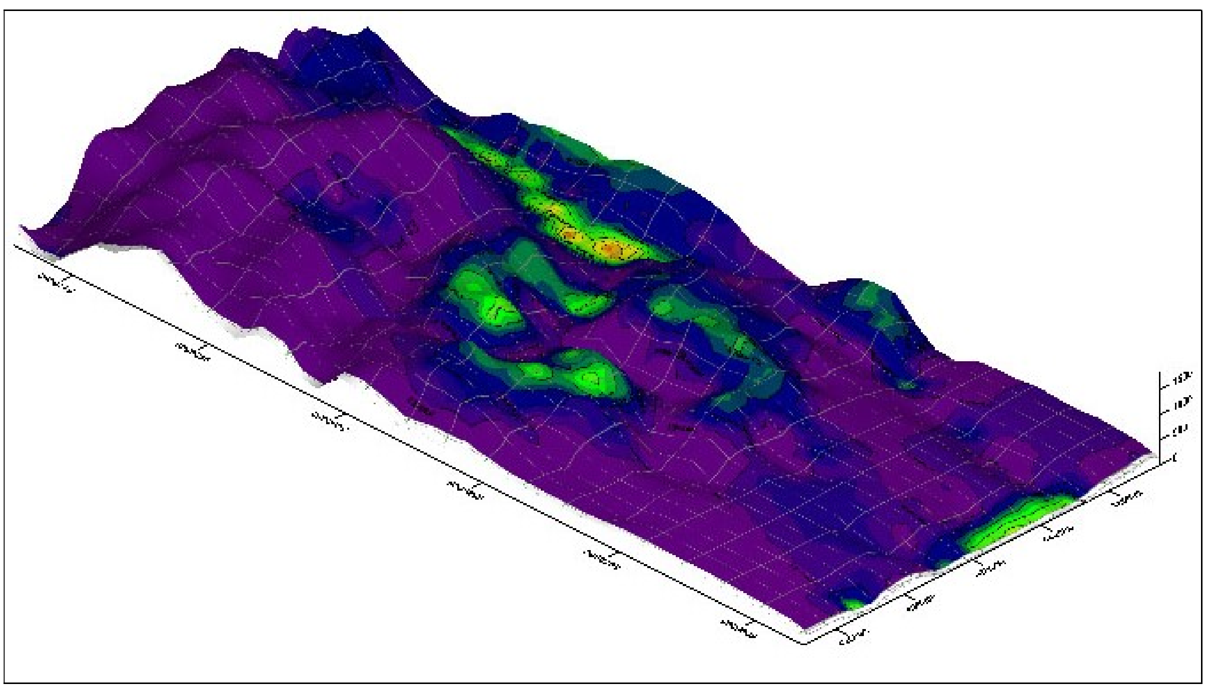


Pollution in the Lecco District by contour lines

Pollution in the Como District by contour lines



Pollution in the Lecco District by 2D ½ DSM

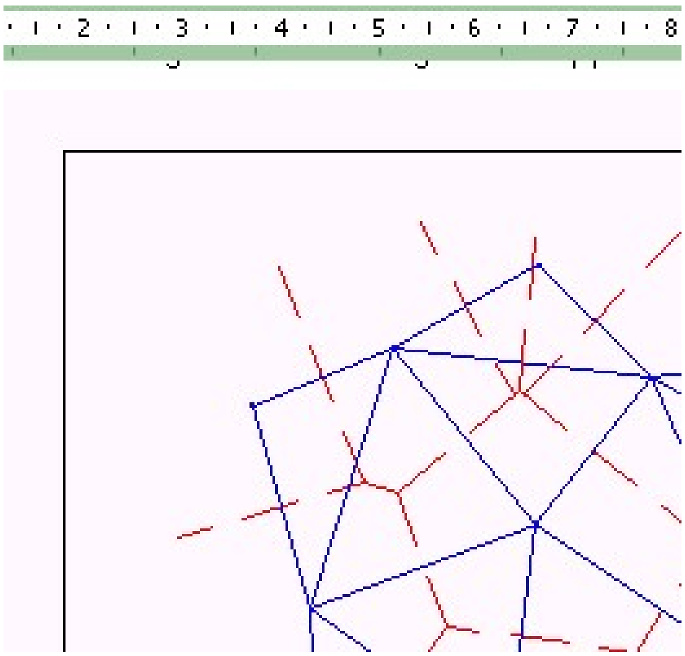


Pollution in the Como District by 2D ½ DSM

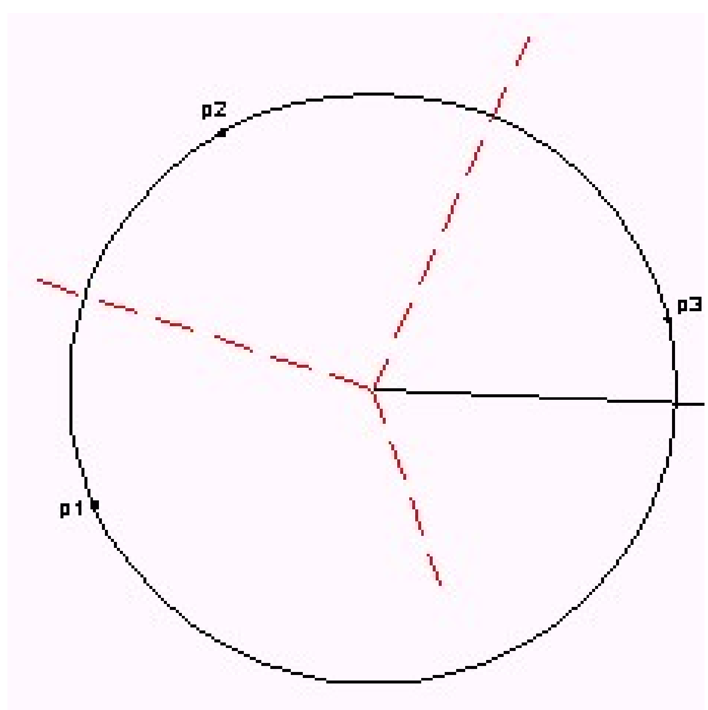
PART II – DSM

Digital Surface Modeling

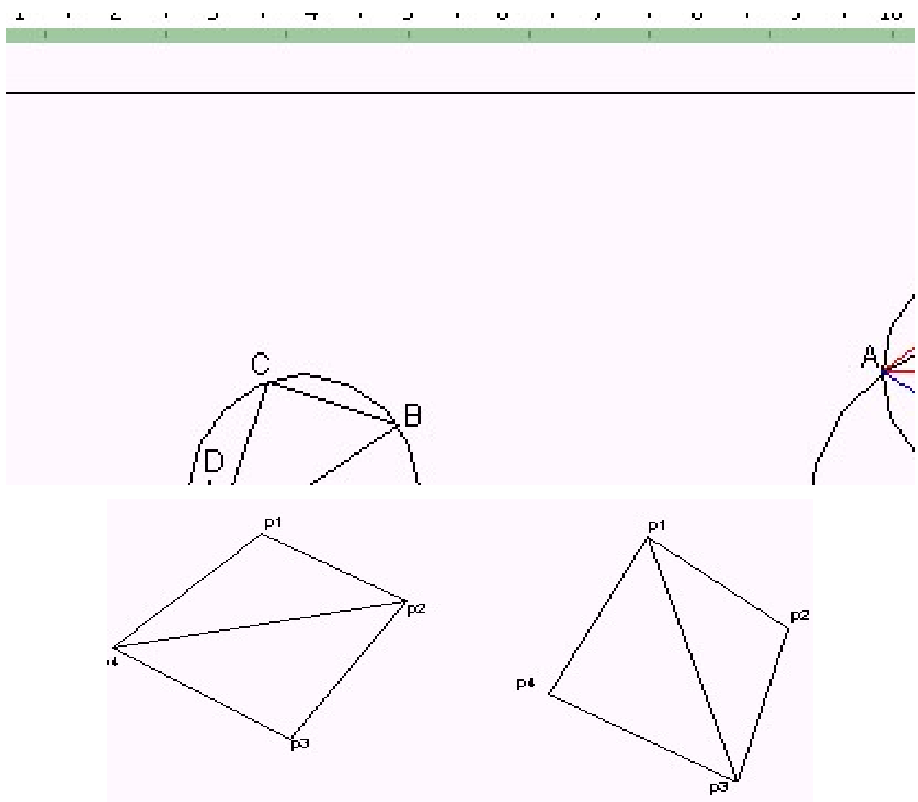
<u>Geometric characteristics</u>



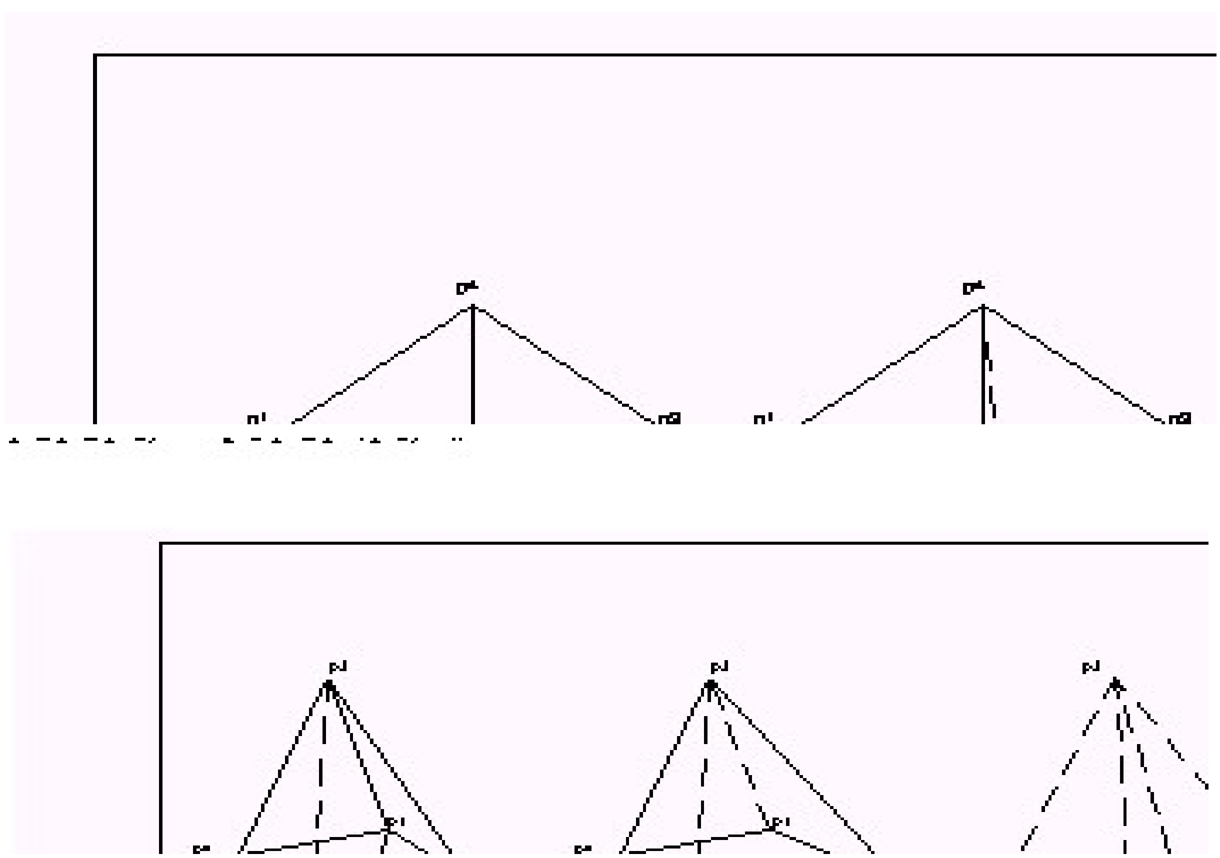
Voronoi diagram and Delaunay triangulation



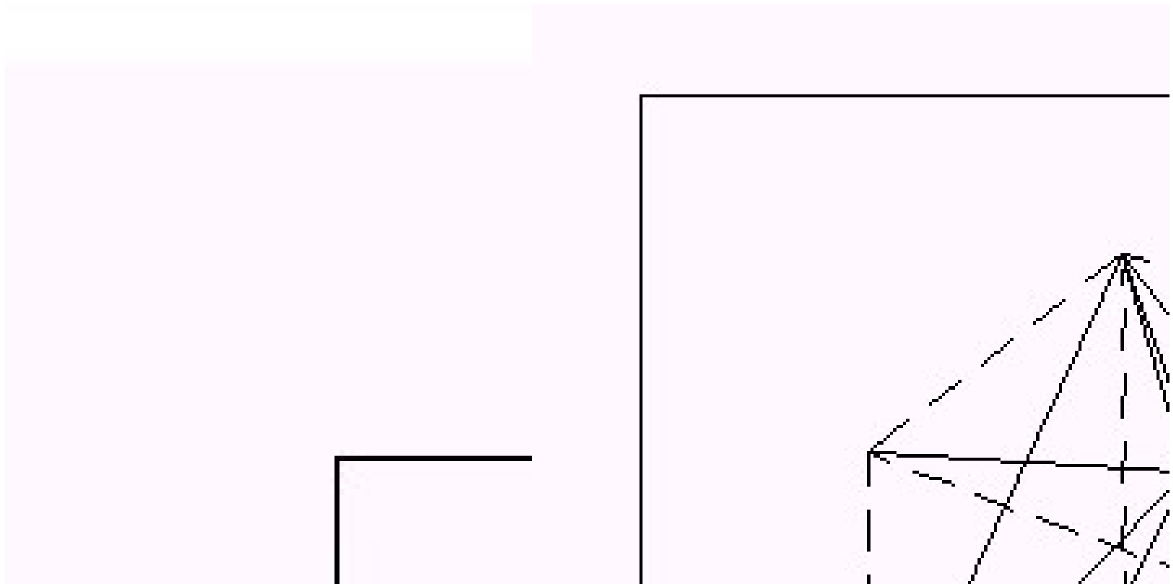
Construction of a Voronoi point



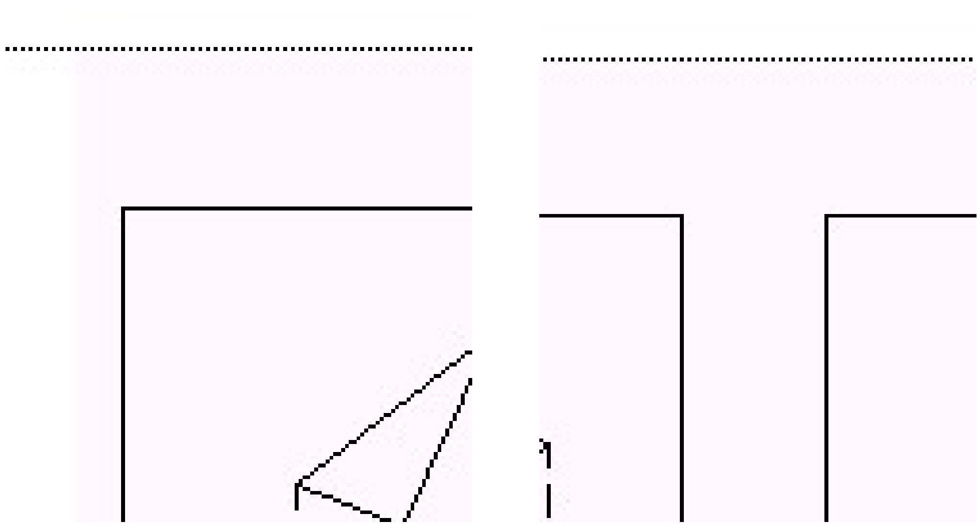
Delaunay criteria



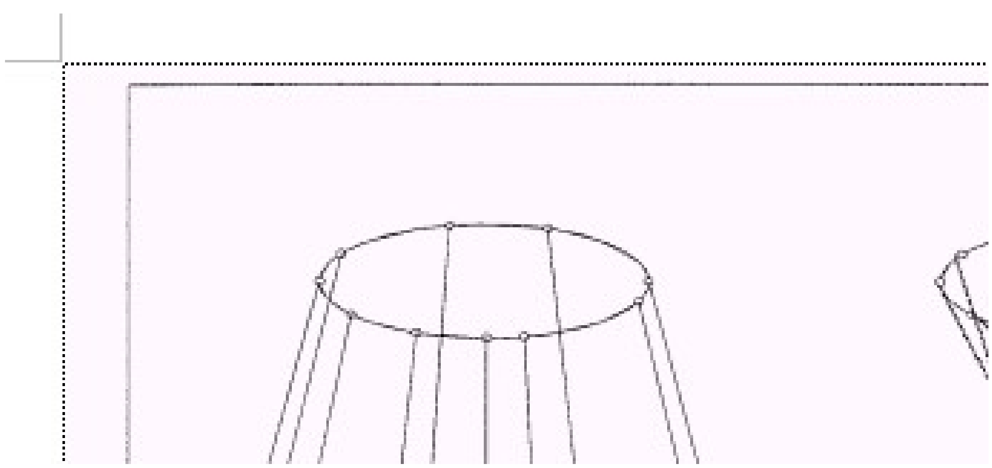
Delaunay 3D triangulation



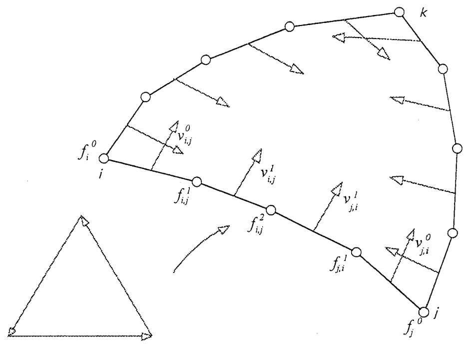
Delaunay 3D criteria

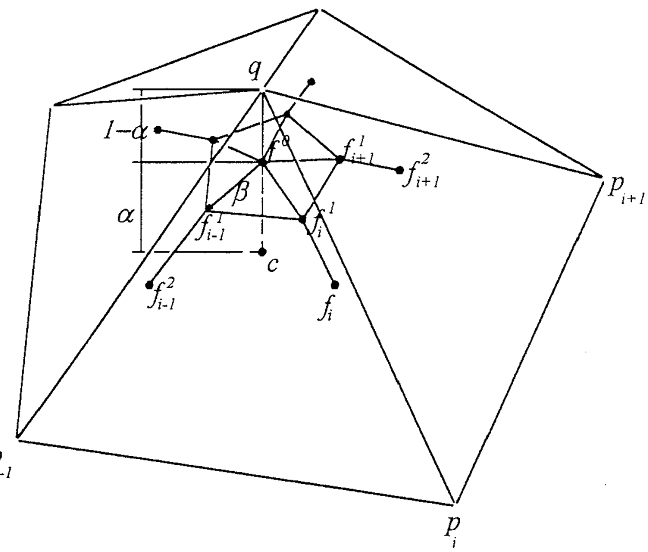


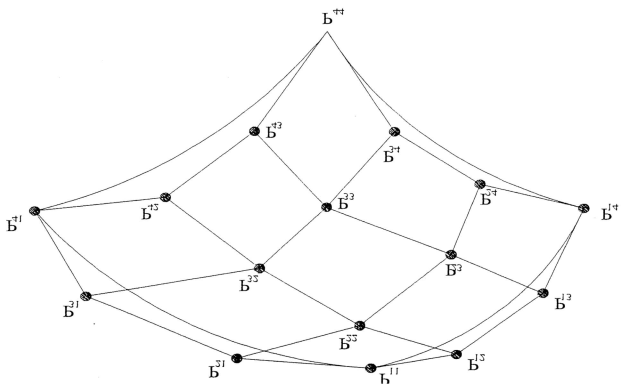
A concave example

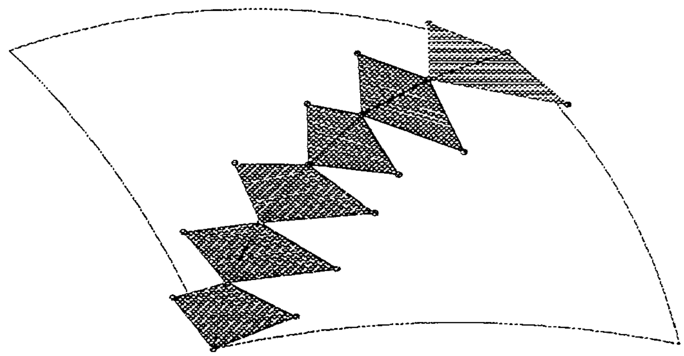


Additional Delaunay 3D criteria





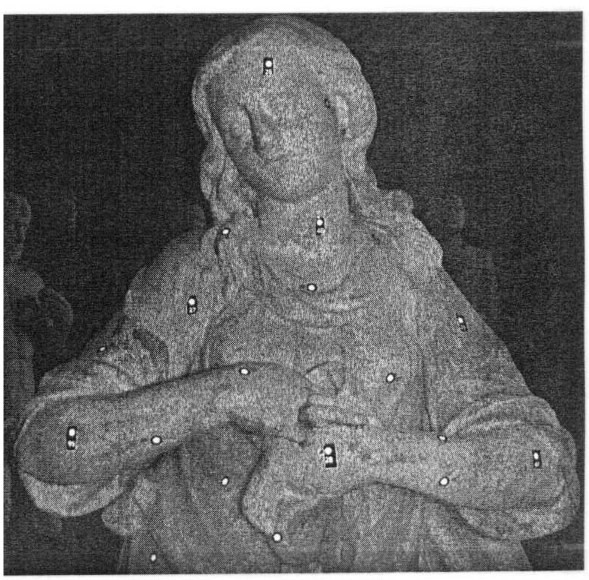


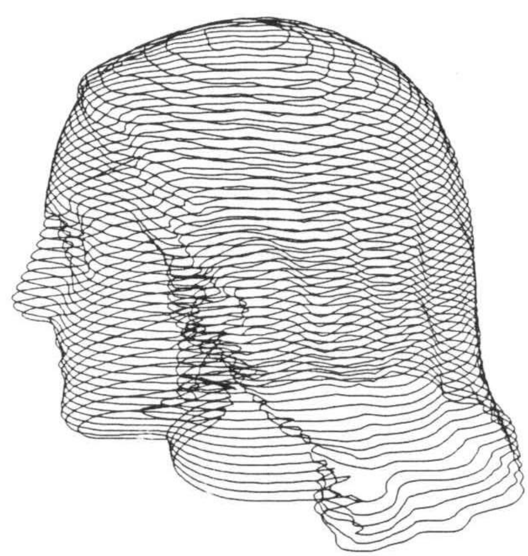


Bèzier triangular spline: definition, construction, testing and assembling

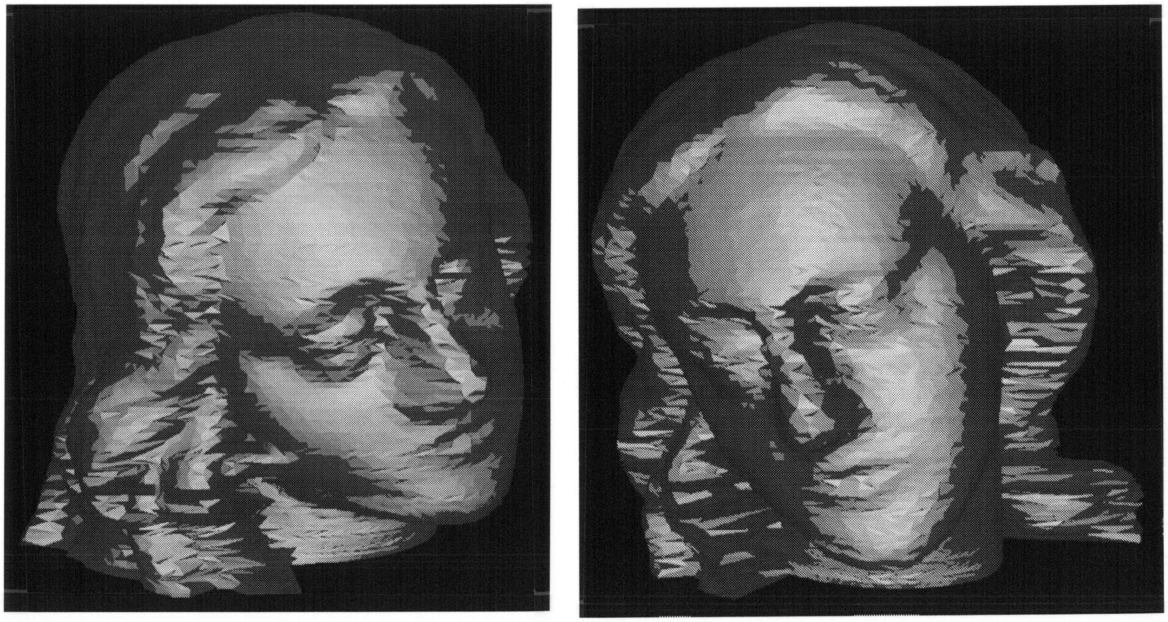
Two examples

Milan Cathedral statue

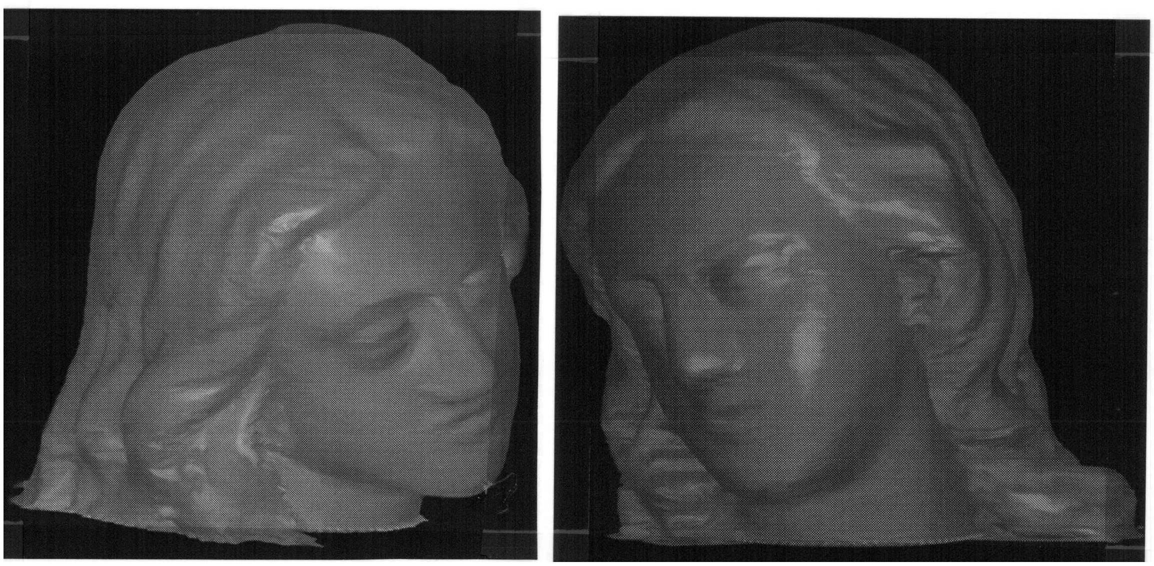




Data acquisition and contour line modeling

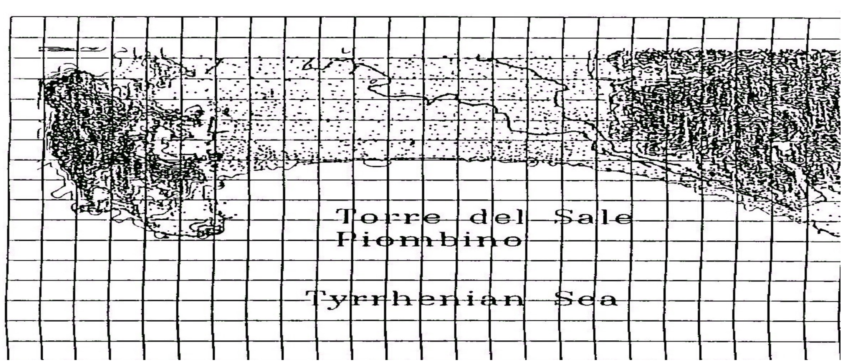


Delaunay 3D triangulation

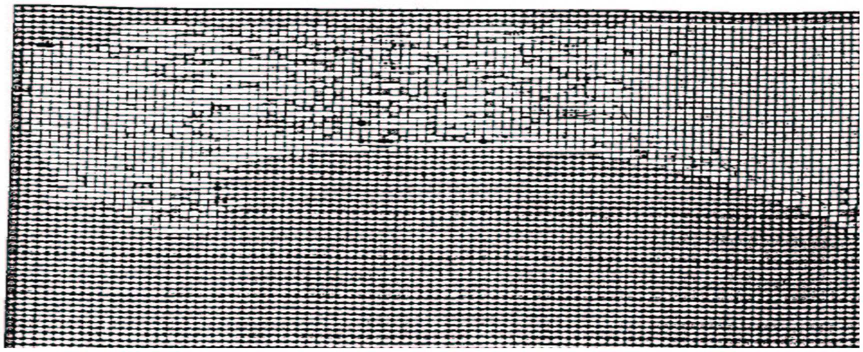


Bèzier triangular spline

Fig. 11 - Rappresentazione a curve di livello confrontata con i dati in ir

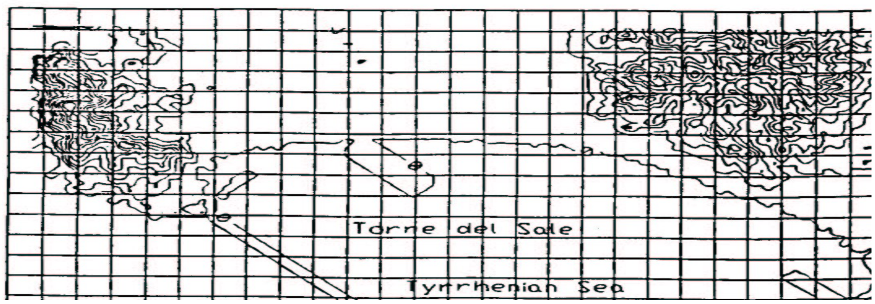


Torre del Sale (Poimbino) area

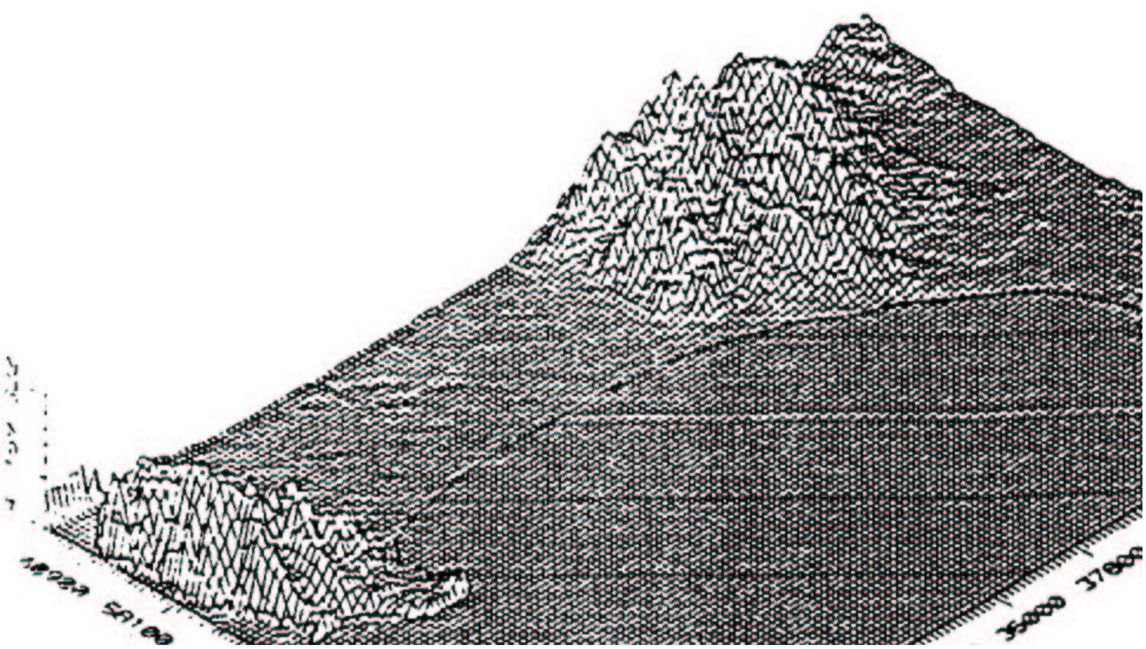


Data validation

La figura 13 mostra un esempio dovuto alla mancanza di punti nel campioname
costiera.



Outlier detection



3D reconstruction

The problem by means of an example

To create a truly good representation of a three-dimensional object one needs mathematical tools. When the object has a complex shape, it's very difficult to obtain its reconstruction, so different methods of interpolation or approximation may be used to generate a Digital Surface Model (DSM) or Triangulated Irregular Network (TIN), according to the shape characteristics of the object. In fact, if the object has a continuous curvature (convex hull) it is possible to generate a 3D triangular surface employing the *Delaunay* algorithm.

Besides, to obtain a good result in terms of visual display it is necessary to apply another algorithm that produces *Bézier* splines on a triangular support. By using this algorithm for the triangulation and smoothing of the generated surface, it is also possible to control the series of points which have to be interpolated and to fix (a priori) the smoothing level of the final surface, with very small fluctuations.

Here a method for automatic reconstruction of models of a piecewise smooth surface is described: some applications will be shown by means of examples. Of course, a realistic reconstruction of the object shape needs a high hardware performance to obtain as a final product the display of the render of the interpolated surface (using CAD software).

The method presented can be divided into steps:

- data acquisition;
- data processing (topological organization);
- interpolation and new data prediction by Catmull-Rom splines;
- selection of an interpolation technique based on the triangulation method (Delaunay algorithm);
- smoothing of the surface by Bézier splines.

<u>Data acquisition and topological organization</u>

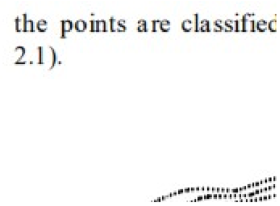
To obtain a DSM that completely describes an object having a complex surface, it is necessary to survey a lot of points. Data acquisition can be obtained by different and mostly automatic technologies: mechanical coordinate machine taster, laser scanning, photogrammetry, etc. Among these techniques, photogrammetry provides the survey of many interesting points with good results. This task unfortunately can't be performed in a short time and without a great effort by human operators using classical photogrammetry.

On the contrary, digital photogrammetry can solve this problem by applying the fast and automatic method of least squares matching on two or multi images of the object, taken from different points of view. It is important, in this kind of survey, to know not only the coordinates of the points, but also their topology. Data acquisition sometimes results in a series of *n* points of the 3D object surface, which are not always organized in a topological way.

In fact if one takes a series of *n* points, lying on the external surface of a 3D object, it is important to reconstruct their topology, before applying the algorithms to interpolate or approximate the object surface. Moreover it is necessary that the whole point set lies on the convex hull, to obtain a digital elevation model close to the object; on the contrary, when there are only few points on the external surface, it obtains a 3D representation is not obtained with an acceptable approximation. Note that there is not only one efficient solution for every possible case.

Data processing, the phase in which the topological structure of the data is fixed, becomes the necessary preliminary step for data reordering, using different criteria.

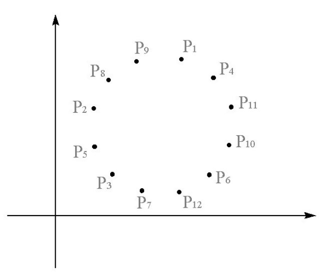
In this work, the series of given points, acquired by photogrammetry, is structured in planar sections (like contour lines or profiles), where the points are classified by different heights.

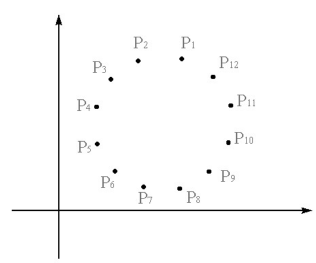


A sample of 3D data structured in contour lines

Topological organization allows for the reordering of the points belonging to the same planar section, researching the relations between the points and then predicting new points.
Scientific literature describes different criteria to this purpose in a short time and with good results. To this aim we decided to apply the *minimum distance criterion*: two points which lie in the same planar section, are neighboring if and only if they are at the *minimum distance* from one another with respect to any other point of the data set.
During this step the new numbering of the points is done too, and the pre-processing of this topological organization is divided into three steps:

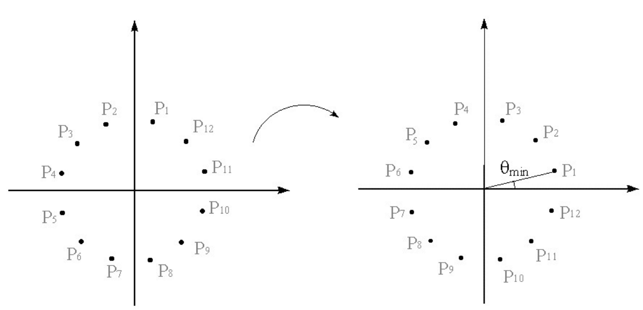
- for every planar section, the coordinates of the center are calculated;
- the reference frame is shifted from the origin to the coordinates of the center;
- for every planar section a new starting point is identified to organize a new numbering using cylindric coordinates with origin in the center.





Original position of the points structured in planar sections

Topological organization by a *minimum distance criterion* and new numbering



Search of a center and shift of the reference frame

New reordering of the points, starting from first direction

The first point is the one coincident with the first direction, so that the new numbering of the series starts from this point with the given step, running counterclockwise.
After data preprocessing it is possible to apply to the reorganized data set interpolating and/or approximating functions which allow for a fast prediction of new data.

<u>Interpolation of curves by parametric functions</u>
The classical interpolation problem involves replacing a complex non-linear function, by a simpler linear function, in such a way that the interpolating function and the given function have the same values at positions corresponding to a prescribed set of points.
In this section we present the interpolation of curves given in parametric form (since this is more common in practice), rather than functions in the classical form. The *parametric representation* for curves overcomes the problems caused by functional or implicit forms.
Indeed parametric curves replace the use of geometric slopes (which may be infinite) with parametric tangent vectors (which are never infinite). The image of an open, closed, half open, finite, or infinite interval under a continuous, locally injective mapping into 2D or 3D space is called a curve.
A curve can be considered as a set of points, with respect to a given origin. These points can be regarded as vectors, which are the values of a locally one-to-one *vector-valued* function of a given parameter defined on above defined interval. This function is called the parametrization of the curve.
Given $n+1$ pairwise distinct points in the 3D space, associated with (appropriately selected) parameters, there are different polynomials which interpolate a curve and different ways to choose the value of this parameter, depending on the final shape of the interpolating curve, on the computing time, on the accuracy required.
We generally prefer the parametrization of curves because it gives us greater flexibility and some advantages. In fact, these functions offer:

- less constraints for the control of the shape, since every component is a function of the selected parameter;
- computational advantages and fast programming, since they use the vector-valued form.

The interpolating functions, which are used in particular for many applications in modelling, are polynomials of low degree $3 \leq m \leq 5$. They describe a given set of empirical data, corresponding to measurements by means of curves with different degrees of smoothness and performed in such a way as to minimize a prescribed error measure and undesirable fluctuations.
So a curve is approximated by a *piecewise polynomial* curve; each segment of the overall curve is given by three functions, which are cubic polynomials in the selected parameter.
Cubic polynomials are most often used, because lower-degree polynomials give too little flexibility in controlling the shape of the curve, while higher-degree polynomials can introduce unwanted wiggles and also require more computations.
Higher-degree curves require more conditions to determine the coefficients and can "wiggle" back and forth in ways that are difficult to control. Higher-degree curves are used in applications in which higher-degree derivatives must be controlled to create surfaces that are aerodynamically efficient. In fact, the mathematical

development for parametric curves and surfaces are often given in terms of an arbitrary degree *m*.
If we fix *m* = 3, three cubic polynomials define three curve segments. To deal with finite segments of the curve, without loss of generality, we restrict the selected parameter to the a zero-one interval. Setting and defining the matrix of the coefficients of the three polynomials, we can rewrite these equations, so that they provide a compact way of the same expressions.

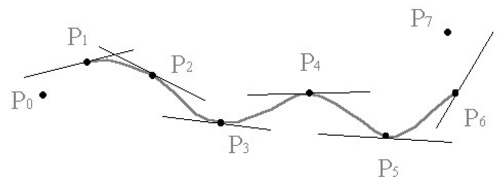
If two curve segments are linked together, the curve has G^0 geometric continuity. If the directions (but not necessarily the magnitudes) of the vectors tangent to the two segments are equal at linkage points, the curve has G^1 geometric continuity. In computer-aided design of objects, G^1 continuities between curve segments are equal at the linkage points. If the tangent vectors of two cubic curve segments are equal (i.e., their directions and magnitudes are equal) at the segments linkage points, the curve has first-degree continuity in the selected parameter, or parametric continuity, and is said to be C^1 continuous.
Each cubic polynomial has four coefficients, so four constraints will be needed, allowing us to formulate four equations in the four unknowns, then solving for the unknowns. The three major types of curves are:

- *Hermite* (defined by two endpoints and two endpoint tangent vectors);
- *Bézier* (defined by two endpoints and two more points that control the endpoint tangent vectors);
- *several kinds of splines* (each defined by four control points).

The different types of parametric cubic curves can be compared using different criteria, such as ease of interactive manipulation, degree of continuity at linkage points, generality and computing time needed.
To see how these coefficients depend on the four constraints, we suitably rewrite their coefficient matrix, using a *basis matrix*, and a four-element column vector of geometric constraints, called the *geometry vector*. The geometric constraints are the conditions, such as endpoints or tangent vectors, that define the curves representing the *cubic polynomials* in the selected parameter.
Very often, we have a set of positions and want a curve to interpolate (pass through) them smoothly. The *Catmull-Rom* family of interpolating or approximating splines, also called *Over-hauser splines*, are useful in this situation. A spline belonging to this family is able to interpolate points in the sequence of points. In addition, the tangent vector to a certain point is parallel to the line connecting this point and its successive. Unfortunately, these splines do not have the convex-hull property. The natural (interpolating) splines also interpolate points, but without the local control guaranteed by the *Catmull-Rom* splines (whose design is given by the *Catmull-Rom* basis matrix and the same geometry vector).
In this work we have chosen the *Catmull-Rom* cubic spline curves because they have the characteristic of allowing for a fast algorithm. This choice was made also to match these requirements:

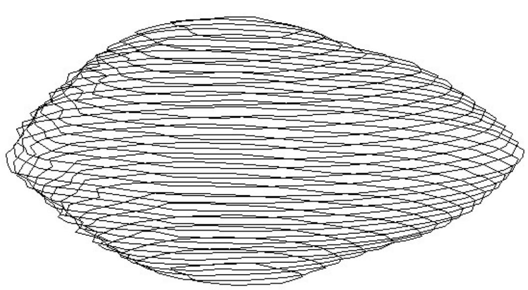
- finding an interpolating smoothing curve with C^1 continuity;
- finding a curve with only local perturbations, without too many modifications on the complete surface;
- doing the change of a given point in the series or predicting a new point without having to completely compute the curve again but only the neighbors to the point.

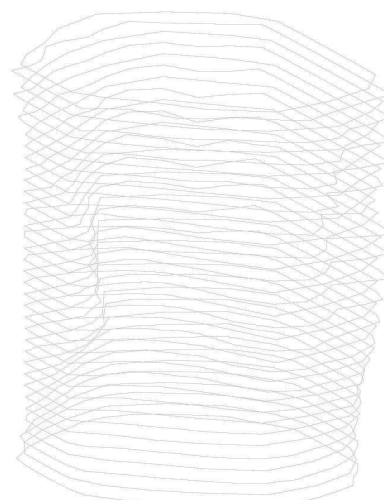
The *Catmull-Rom* cubic spline curves interpolate the point set, in the sequence organized by the preceeding topological criterion. In fact, every segment of this curve passes through each point in a parallel direction to the line between the adjacent points with continuous curvature C[1]. The straight-line segments indicate these directions.



The spline passes through each point in a direction parallel to the line between the adjacent points

The choice of the parameter is free but depends on the final shape that we want to obtain. In the algorithm implemented and described in this work, we have used an equally spaced parametrization. The prediction of a new point is easy, since we introduce the parametric value *t* of this new point in the *Catmull-Rom* equation using four points around it. The introduction of a change in the position of a point causes a deviation only in the four segments of curve neighboring this point. Therefore this type of curve is only locally disturbed. The algorithm implemented uses the series of points previously organized to predict new points on each section, necessary to correctly apply, later on, the *Delaunay* triangulation according to the final complex surface representation (as shown in the two following examples).





First example of interpolation by *Catmull-Rom* splines

Second example of interpolation by *Catmull-Rom* splines

Note that one must multiply the number of points by a coefficient one for curves of odd order and two for curves of even order. This procedure will be useful when the *Delaunay* triangulation algorithm will be exploited to research the connections among the points of different planar sections.

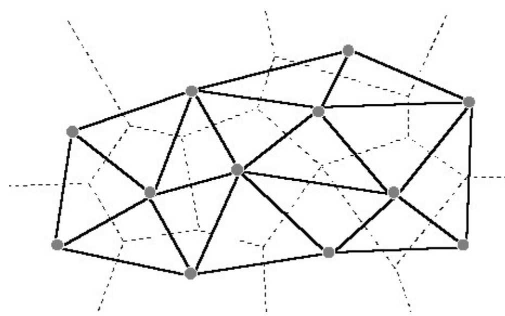
Application of the *Delaunay* triangulation algorithm

In this section, we discuss a method, based on the triangulation reconstruction, of the convex-hull of the data point set, lying on the planar sections, where the vertices of the triangulation coincide with the given points.

For each triangle, we construct a surface patch which interpolates the given function values (and possibly also the derivatives) at the vertices. There are a number of methods available and they can be differently combined.

The need to construct a globally optimal triangulation suggests that we work with the *d*-dimensional version of the *Delaunay* triangulation (the straight-line dual of the *Voronoi* tessellation). An appropriate triangulation is generally chosen such as to satisfy some optimality criterion which guarantees, first of all, a unique triangulation, possibly without elongated triangles.

A globally optimal triangulation (that is, of course, locally optimal) is the triangulation associated with the *max-min angle criterion*. To explore this, we recall that given a point set, the corresponding *Dirichlet* tessellation (also called the *Thiessen* or *Voronoi* tessellation) is defined as the partition of 3D space into *Dirichlet* tiles.



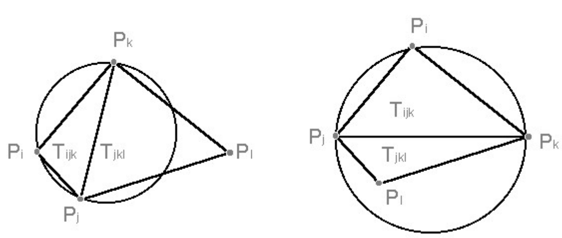
Voronoi tessellation and associated *Delaunay* triangulation

By using the *Euclidean* distances, a polygon consisting of all points which are closer to a given point than to any other different point, so that two polygons are pairwise disjoint and all the polygons cover the entire 2D space. Given a point a corresponding *Dirichlet* tessellation can be constructed by finding the perpendicular bisectors to the line segments connecting the various neighbor points.

The *Delaunay* triangulation of a given points is the dual of the *Dirichlet* tessellation; two points are connected if and only if the two corresponding tiles of the associated *Dirichlet* tessellation share a common edge. The *Delaunay* triangulation can be constructed using an appropriate *circle criterion* in 2D dimension or the *spherical circumscribed circle criterion* in 3D dimension. For example, the local circle criterion is satisfied for a quadrilateral with vertices provided that the circumscribed circle associated with a first triangle with three vertices (selected among the four of the quadrilateral) does not contain the forth vertex of the second triangle with three vertices, defined by this forth vertex and by the vertices which share the common edge.

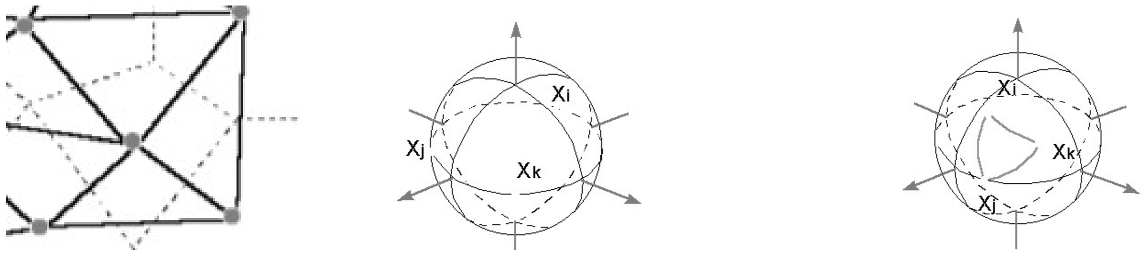
If the *local circle criterion* is satisfied for every convex quadrilateral, so is the *strong global circle criterion* which requires that, for every triangle in the triangulation, the associated circumscribed circle contains no other data point.

(a)



(b) Local circle criterion: (a) satisfied, (b) not satisfied

It is also possible to construct triangulations on curved surfaces using curved triangles (e.g., spherical triangles on the surface of a sphere). In fact the *spherical circumscribed circle criterion* checks whether a given point lies inside or outside the spherical circle passing through the three other points.

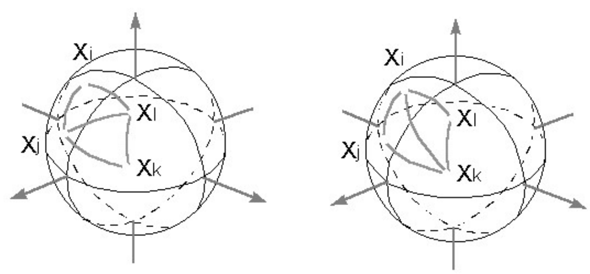


Not admissible

Admissible

Not admissible spherical triangles and admissible spherical triangles

A triangulation method can be applied to general convex surfaces. Given a set of points, we can define a 3D triangulation, in direct analogy with the definitions reported above also if there are essential differences between triangulations in the plane and those in 3D space which must be underlined.

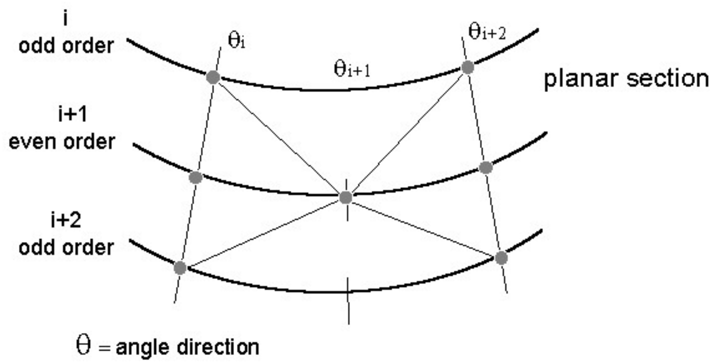


Neighboring triangles with two different diagonals

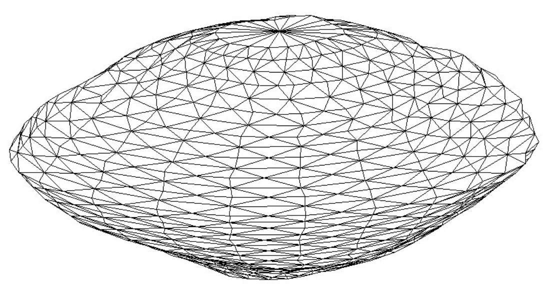
For example, in the plane the set of points and its convex-hull uniquely determine the number of triangles and the number of edges of the triangulation. In a higher dimension this is not so simple. So, in the 3D space we cannot always distinguish triangulation on the basis of which data points are connected to each other. Moreover, an iterative construction of the triangulation is not always possible in the three-dimensional case. While the *max-min angle criterion* cannot be directly generalized in the 3D space, we can construct the *Delaunay* triangulation by using a version of the *circumscribed circle criterion* involving (hyper) spheres.

We want now to illustrate the method to realize a 3D triangulation of a set of points. In the last section we have shown the topological and interpolating criteria to organize the data. In this phase we search the correspondence among points which belong to planar sections which are next to one another. This is important in order to correctly obtain the final triangulated surface; in fact, a wrong correspondence can produce a false interpretation of the model of the complex surface. Among the different methods tested to search corresponding points lying on neighboring contour lines, we have chosen the *"direction criterion"*. This method is simple and fast also for complex surfaces. It consists in connecting the points, expressed in polar coordinates, of planar sections of different order, which have the same angle direction.

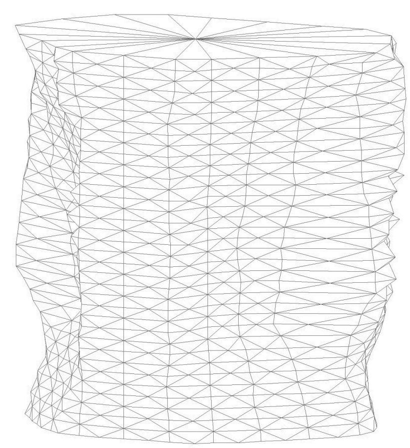
In this way it is simple to construct the 3D triangulation, because one needs to connect N points predicted on a planar section of odd order with 2N points predicted on a planar section of even order; we recall that the points must have the same angle direction. Note that in this case we must satisfy the above mentioned *circle-criterion* of *Delaunay* (as shown in the following two examples).



Planar sections of different order: the points with the same angle direction have been connected



First example of *Delaunay* triangulation



Second example of *Delaunay* triangulation

<u>Smoothing of the surface by *Bézier* splines</u>

Now we will describe a type of surface representation by triangular splines on the *Delaunay* triangulation. Afterwards we will show examples of construction and representation of three-dimensional objects closed by a series of surveyed points, obtained fitting the surface constituted by triangular patches. An interpolating method using global continuous C^r piecewise polynomial functions is defined using the triangular mesh as starting point.

The last phase of the surface reconstruction produces a new mesh optimization by exploiting the *Bézier* method. This method, although even if it is locally approximated, allows to obtain good results in terms of visual display. A triangular control mesh is approximated by a piecewise C^1 continuity spline surface composed by sestic triangular *Bézier* patches.

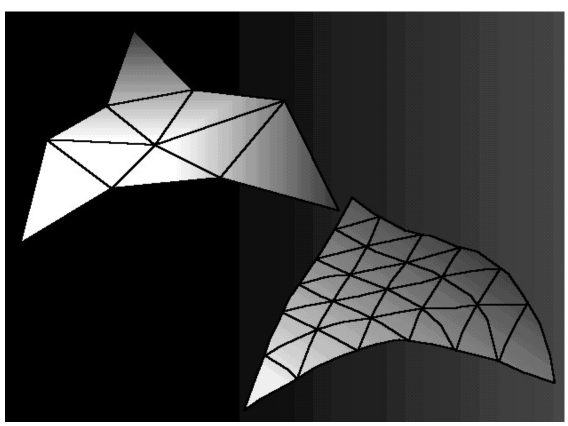
Modelling of the three-dimensional objects can be obtained through elements (patches) of limited dimension, geometrically simple, easy representable with simple mathematical functions. Every element is formed by many points, whose coordinates are given by continuous parametric functions in two variables, defined in the limited interval, ranging from zero to one. The choice of the type of patch (triangular, quadrilateral, etc.) and the shape of its sides depends on the chosen method of interpolation. For example, if the *Delaunay* triangulation is used, every patch is one of the triangles. If instead, an interpolation with polynomial functions is used, the sides will be curvilinear.

In order to describe such a surface, the parametric equation of the curves is extended to the bidimensional case, and if we decide that the geometric vector is, instead of constant (as in the case of parametric curves) variable as a function of one of the two selected parameters, the parametric surface of the element is obtained. First for notation convenience, we exchange the two selected parameters and then we allow the points in the geometric vector to vary in 3D along some path which is parameterized on the first selected parameter.

Thus the geometric vector becomes a matrix, then there are cubic polynomial functions and the cubic parametric patches surface is obtained. Indeed we have obtained an equation which shows the dependency on the two selected parameters, and we can again isolate a geometric vector, which is a constant:

The bicubical surface of *Bézier* with regular shape is a surface constituted by rectangular patches, where the geometric matrix consists of 16 control points (or points of *Bézier*) They are the 16 points that define the polyhedron characteristic, and i.e. the patches of the surface of *Bézier.* They control the slope of the boundary curves and the torsions along the boundary curves.

In many practical applications, when the data are not acquired on a rectangular regular mesh, but they represent a series of scattered points, the choice of a patch of rectangular shape is not convenient since usually triangulation techniques are applied for the construction of the shape. In the examples illustrated in this work the triangular surface patch is considered since it is the more natural choice starting from a triangulation of the points.



From the triangular mesh to the *Bézier* patches

To analyze this problem in detail, it is convenient to introduce the *Bernstein* polynomials, associated with a base triangle, to construct the parametric equation of the surface of triangular *Bézier* splines of degree $m=6$, using the 16 points of *Bézier*.

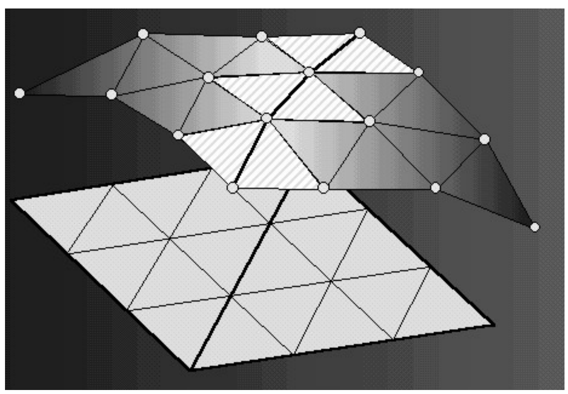
These data form the *Bézier* net, or *Bézier* polyhedron, associated with the surface. It immediately follows that triangular *Bézier* surfaces have the convex-hull properties. The convexity of *Bézier* surfaces is not so easy to be determined that of curves. This is a consequence of the fact that a surface with all convex parametric lines is not necessarily convex.

Since for the construction of the spline of *Bézier* the coordinates of the control points are required and only three of them are defined from the initial data (being the vertices of a triangle coming from a previous triangulation of the series of data), the other ones must be obtained using proper criteria, as functions of the coordinates of the vertices of the adjacent triangles to the one taken into account.

An effective method that allows to determine the points of *Bézier* has been proposed by Loop. This method considers the surface approximating the triangles, whose vertices are the 28 points of *Bézier*. The vertices are computed starting from every side, taking into account the continuity C^1 with the adjacent sides. The method requires to determine four points on every side of the triangular patch and again four points on every segment obtained. Also the direction of the plane of the triangular side to which it belongs is needed.

Therefore the *Bézier* points completely determine the *Bézier* surface, and are also affinity invariantly related to the surface. The *Bézier* net associated with a repeated subdivision converges to the *Bézier* surface. The *Bézier* net approximates the surface, and can be used to compute intersection curves of the *Bézier* surface.

If a single *Bézier* surface is not able to approximate a given surface well enough, then we may use several *Bézier* surface patches which are joined together under prescribed continuity conditions. For example, we can require that visual C^1 continuity implies geometric C^1 continuity.



C^1 continuity for the triangular *Bézier* patches

Different cases of continuity must be taken into consideration:

- the first derivatives coincide along and across the common boundary curve between two Bézier patches (C^1 continuity);
- the first derivatives coincide along the common boundary curve, and the cross derivatives along the boundary curve have the same direction (visual C^1 continuity);
- the two neighboring Bézier patches have the same tangent planes along the common boundary curve (geometric C^1 continuity).

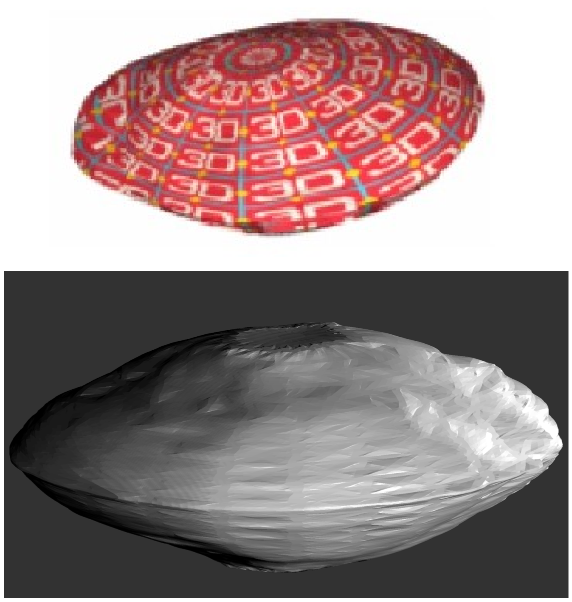
The parametric continuity C^1 between two patches imposes that the control points of *Bézier* lying on the common side and the neighboring points are coplanar. In order to guarantee the continuity of the whole surface, besides the continuity conditions between two adjacent elements, it is necessary to guarantee again the continuity condition of all the elements that meet at a vertex. Once solved the continuity conditions of the patch, one has to impose conditions in order to prevent superimpositions of the same patch and to define the normal vector at every point of the common side.

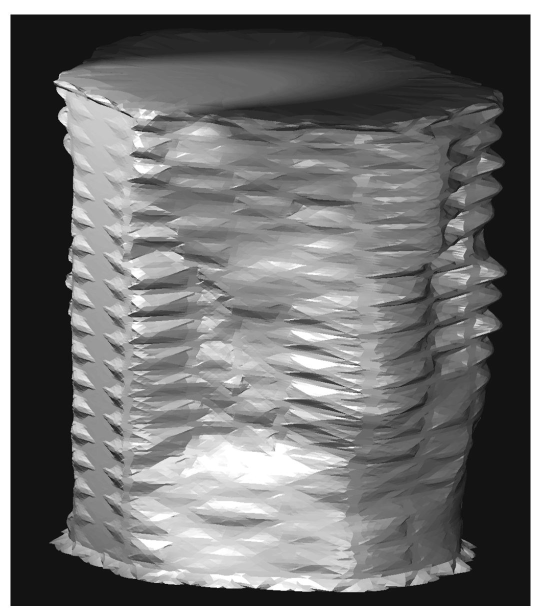
By combining several steps of the algorithm implemented, we can subdivide a triangular *Bézier* patch into an arbitrary number of subtriangles of degree *m*. The sequence of piecewise linear surfaces interpolating the *Bézier* nets converges to the *Bézier* surface.

First example of smoothing surface by *Bézier* splines

Second example of smoothing surface by *Bézier* splines

Usually in designing curves and surfaces, we not only want a good approximation of the data, but also we want the curves or surfaces to be *"visually pleasing"*, in some functional or aesthetic way. In the last section of the paper we have described the procedure to generate a good visual final product. Here we present a realistic reconstruction of natural object using CAD software.

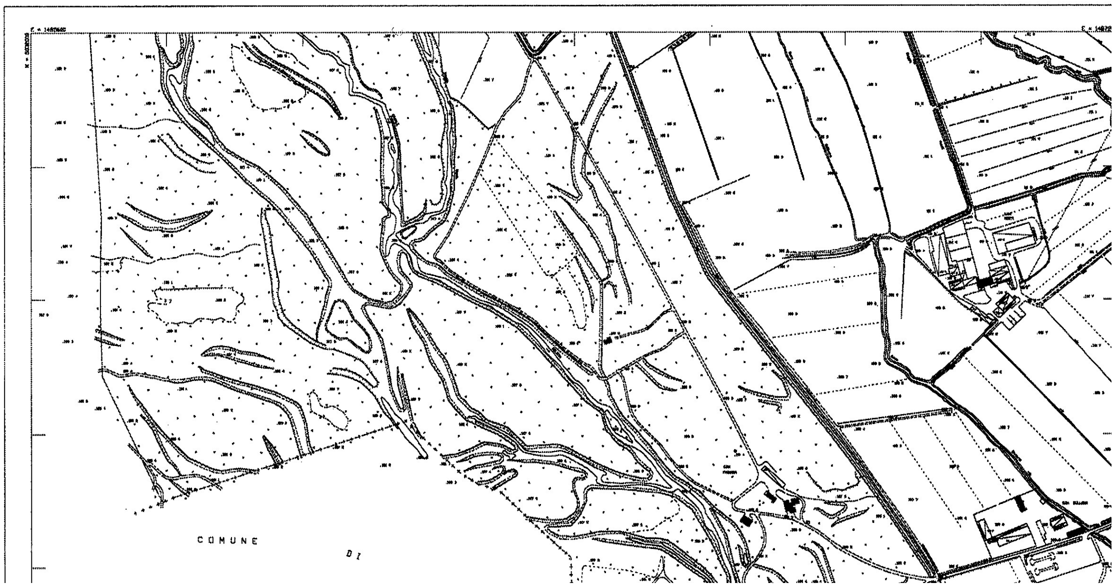




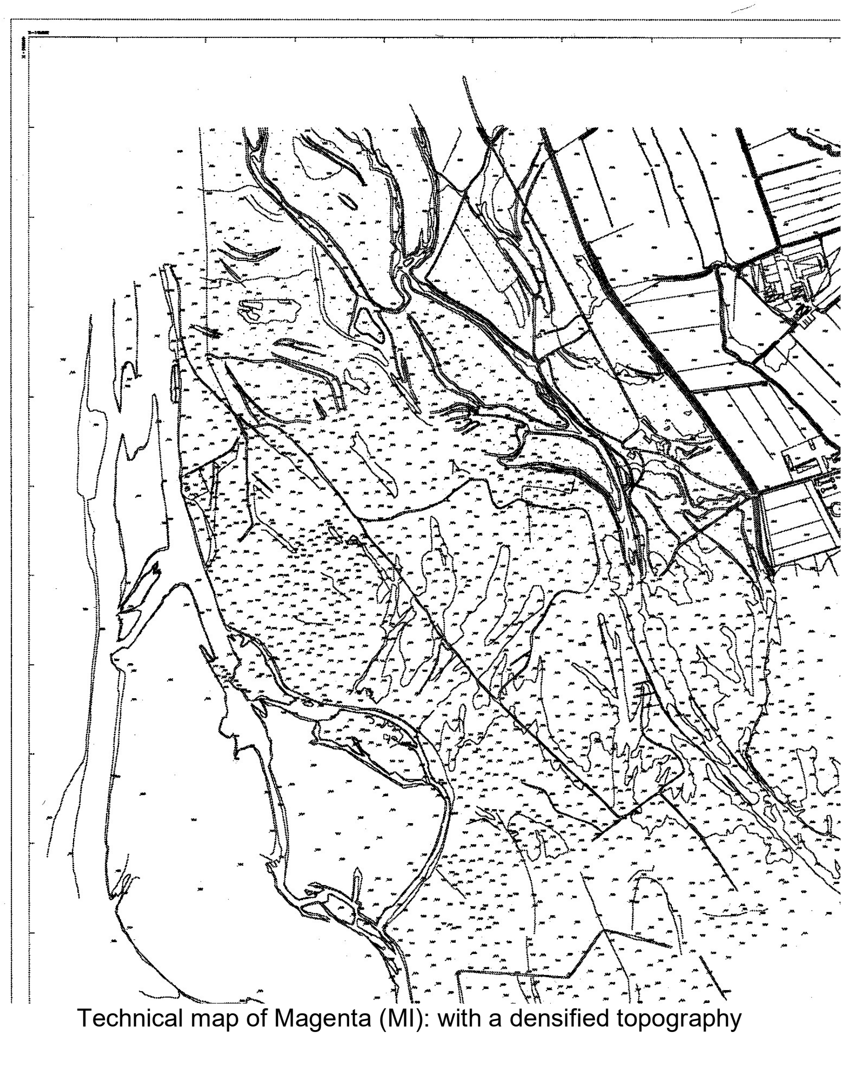
First example: rendering, with a realistic visual display

Second example: rendering, with a realistic visual display

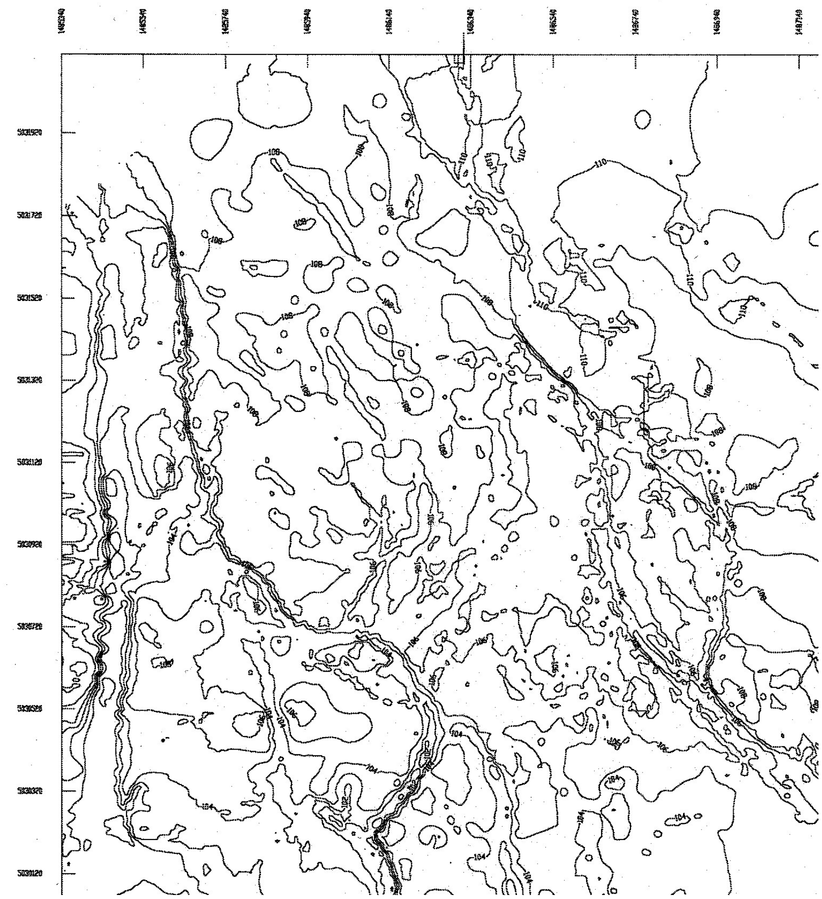
Appendix D – Thematic mapping: two examples

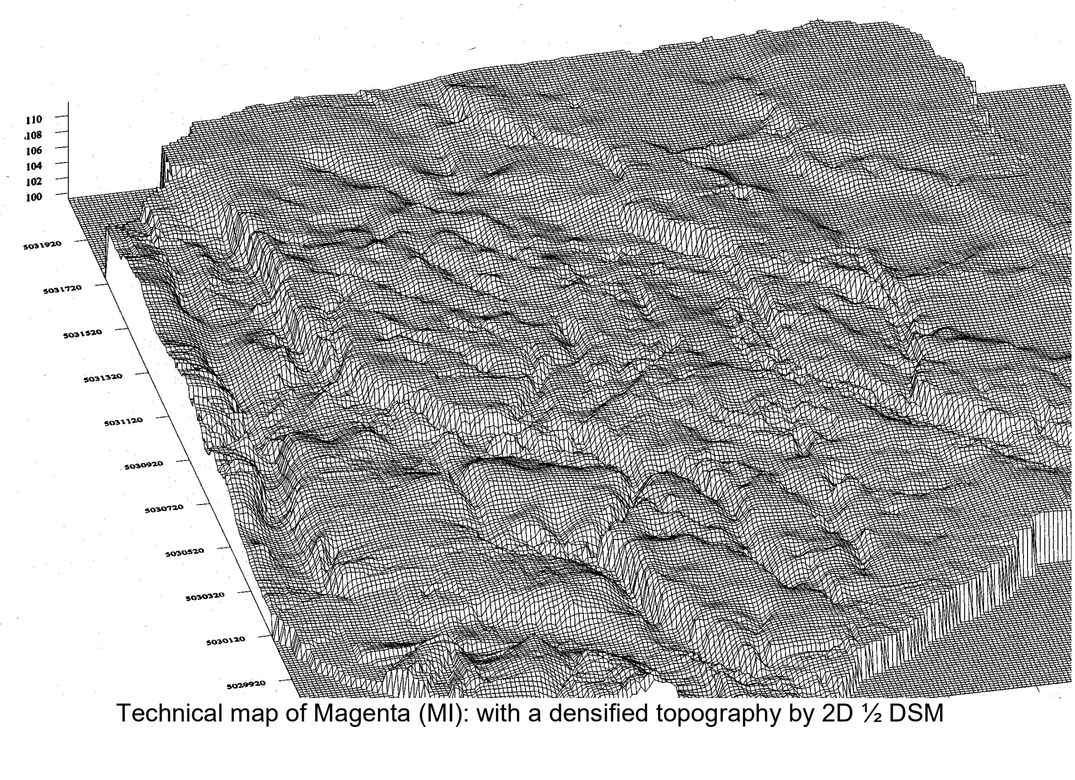


Technical map of Magenta (MI)

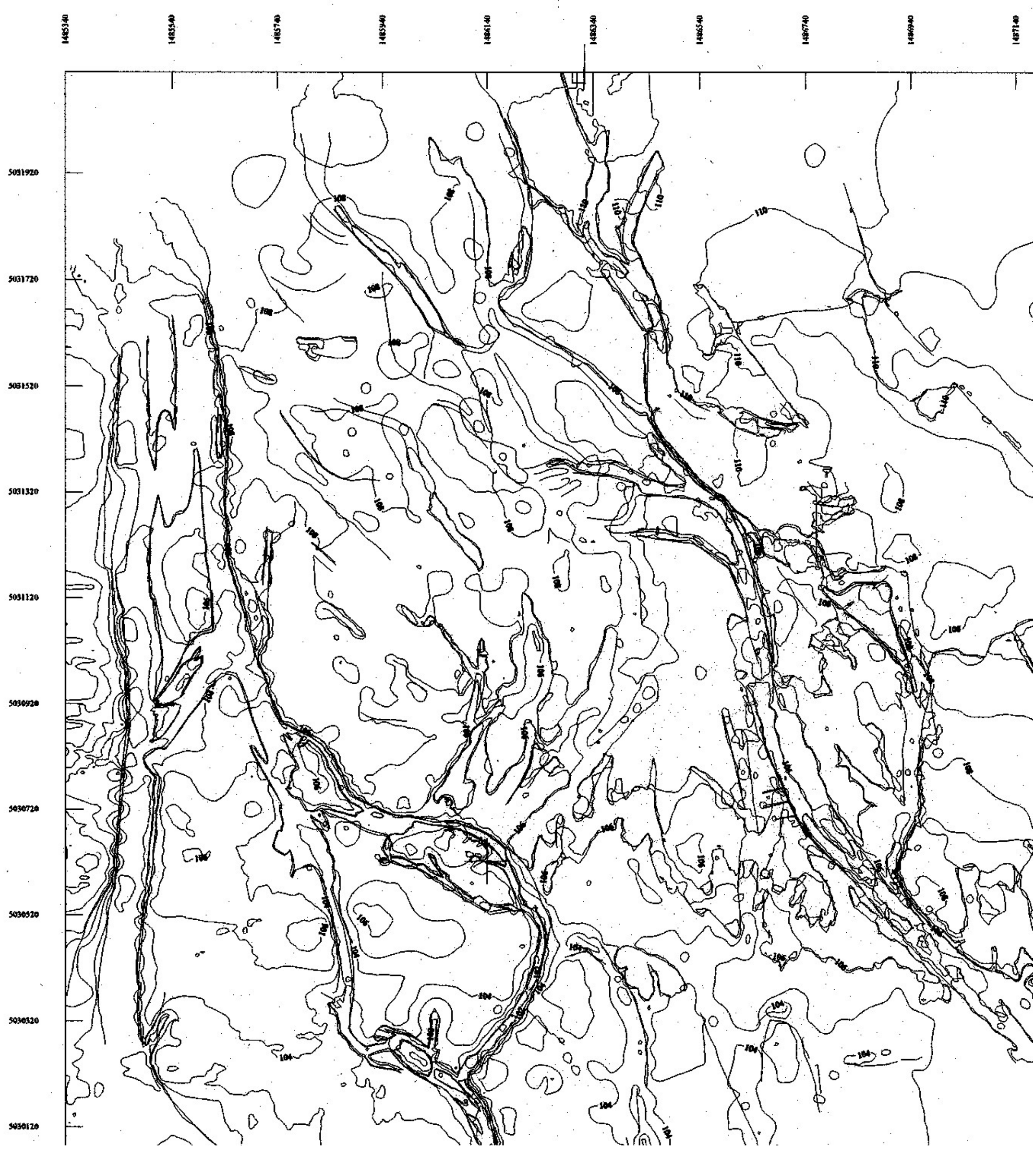


Technical map of Magenta (MI): with a densified topography


Technical map of Magenta (MI): with a densified topography by contour lines



Technical map of Magenta (MI): with a densified topography by 2D ½ DSM

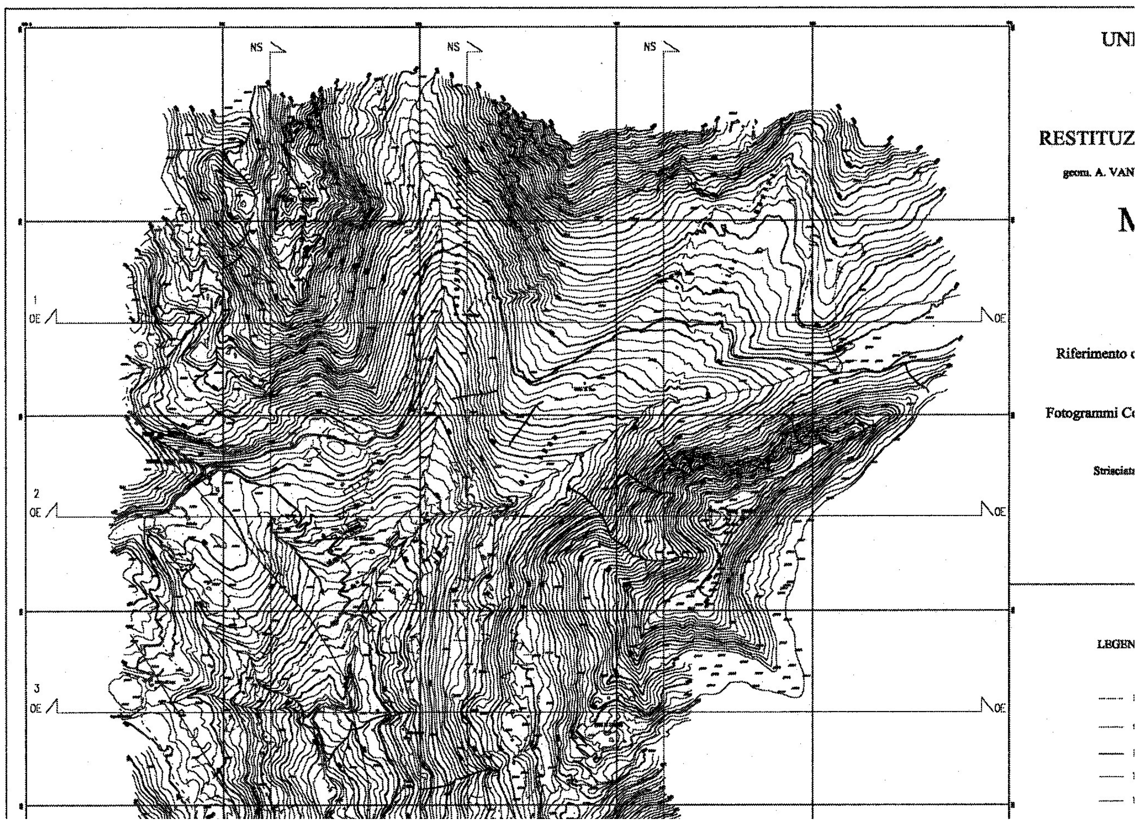


UpdatedtTechnical map of Magenta (MI)

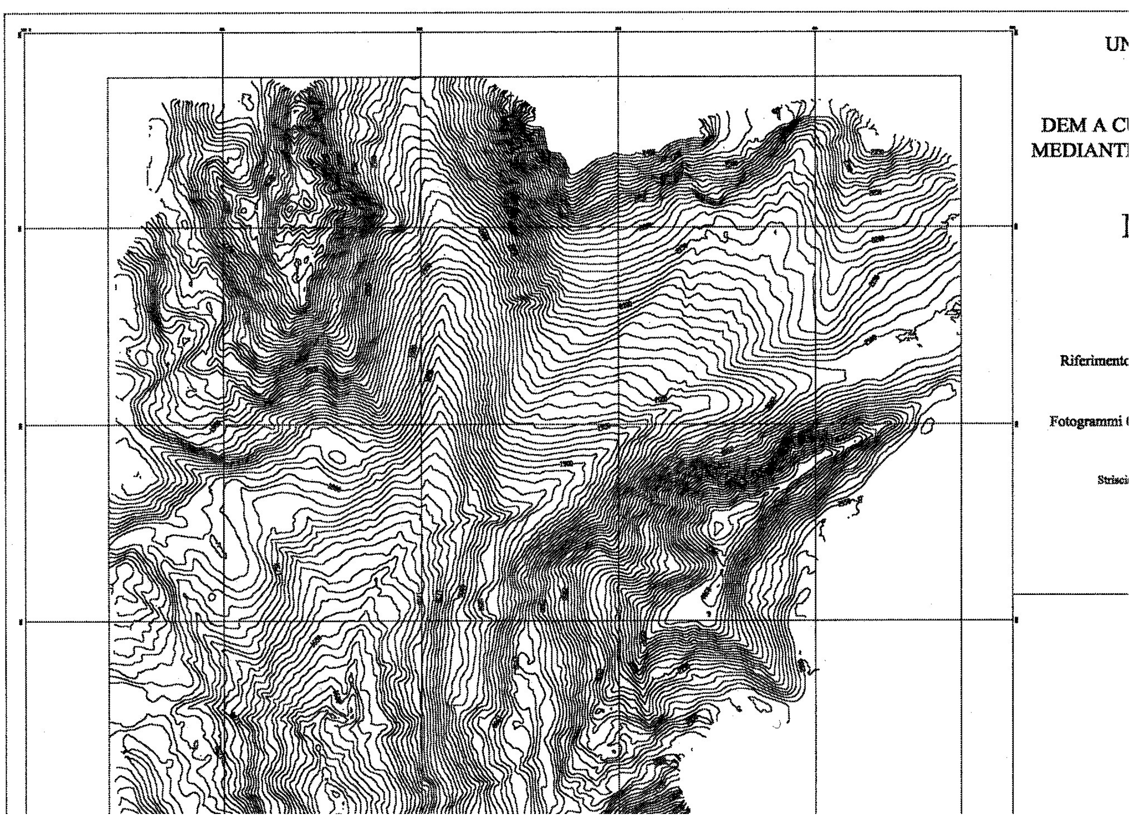
Principal steps [2]:

- image acquisition and block adjustment;
- DSM derivation and data modeling;
- DSM post-processing and visualization;
- orthoimage formation;
- superimposition of vector elements.

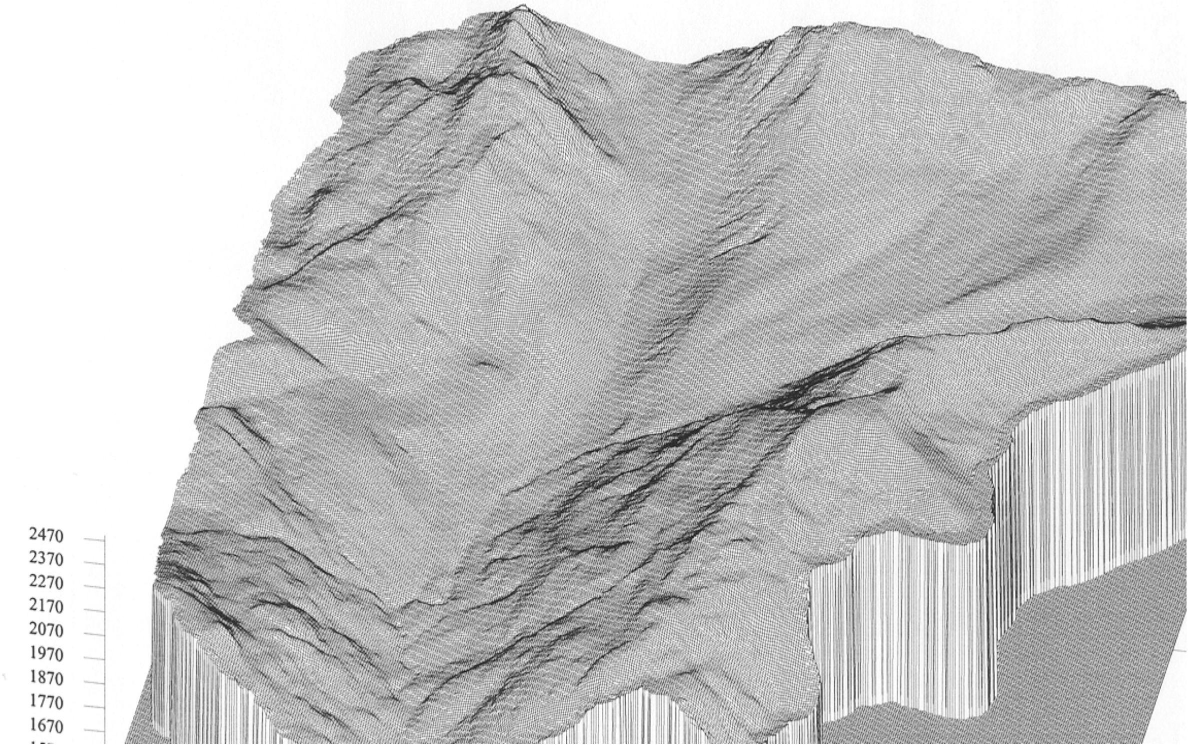
2 They involve methodologies and procedures of digital photogrammetry and computer aided carthography.



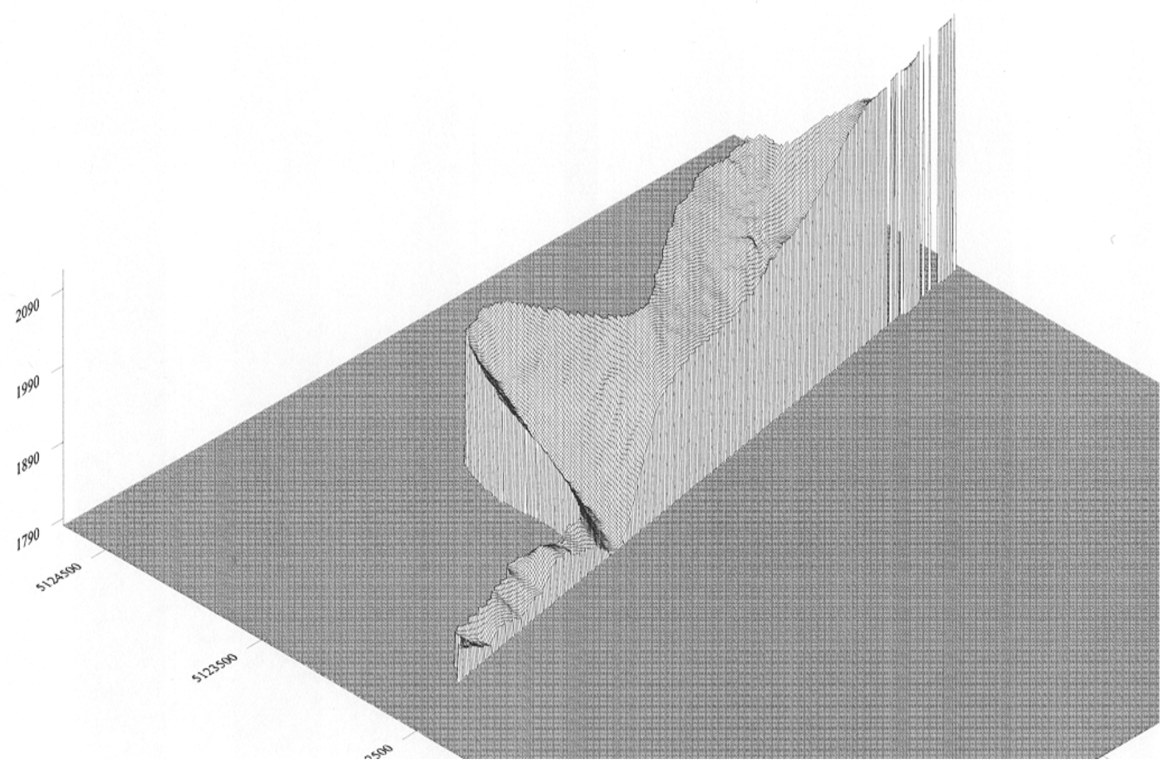
Technical map of Mortirolo (BS) by contour lines



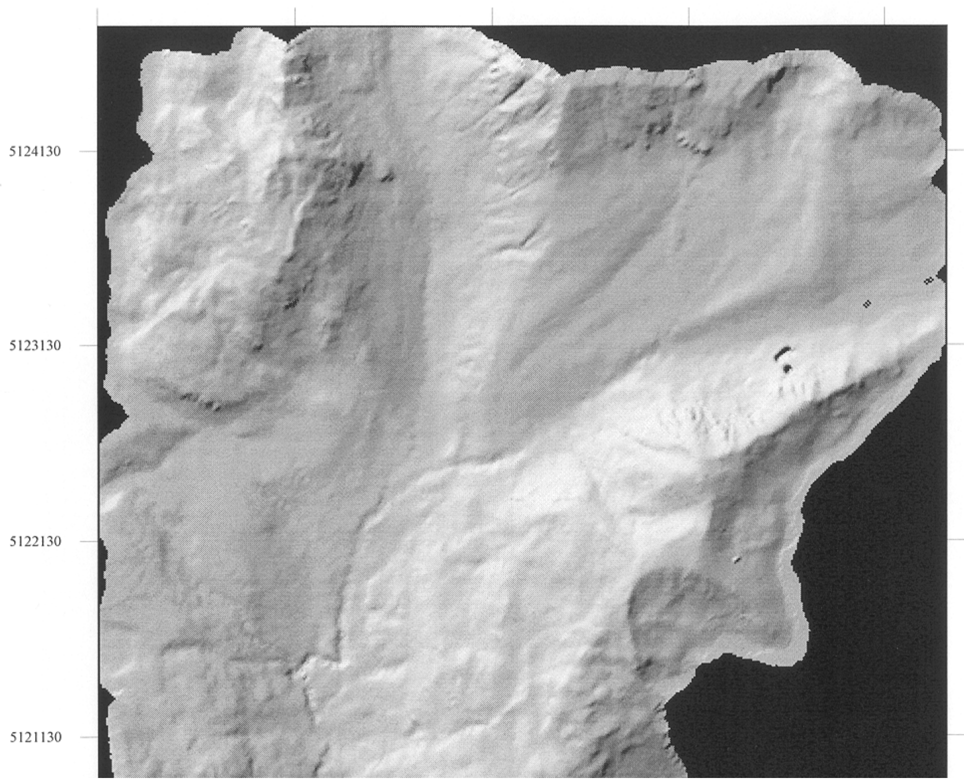
Updated technical map of Mortirolo (BS) by contour lines



Updated technical map of Mortirolo (BS) by 2D ½ DSM



A street profile in the updated technical map of Mortirolo (BS) by 2D ½ DSM



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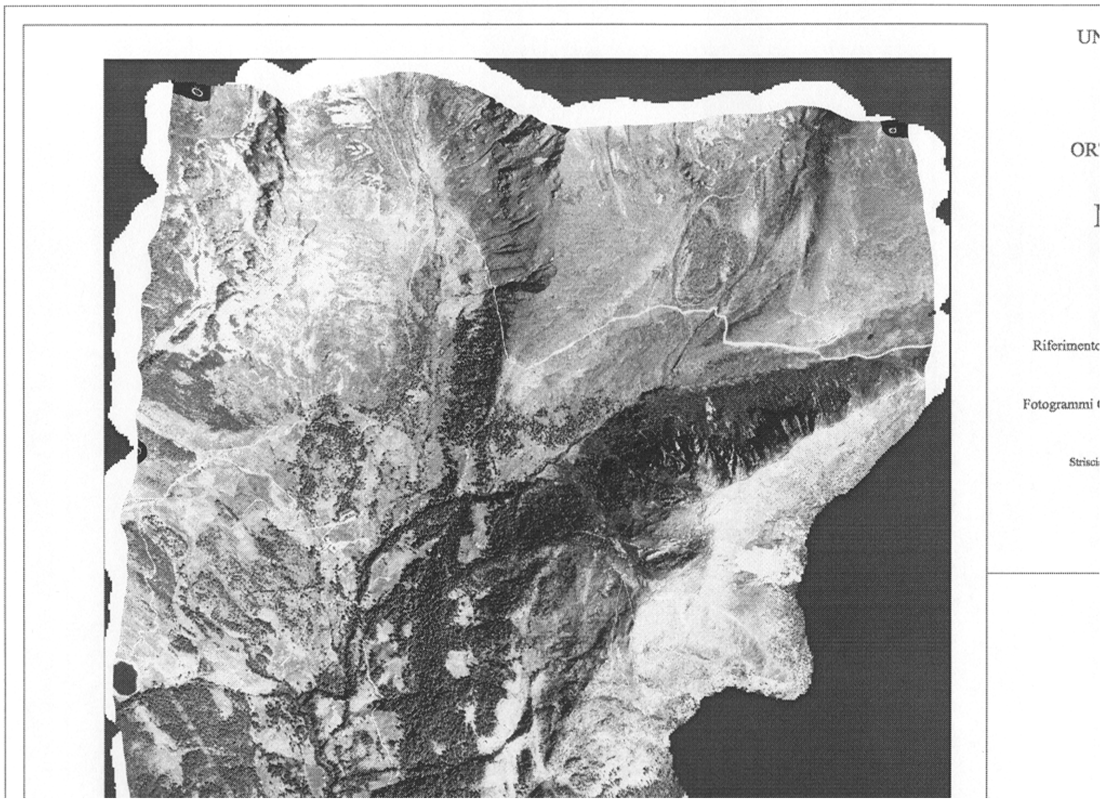
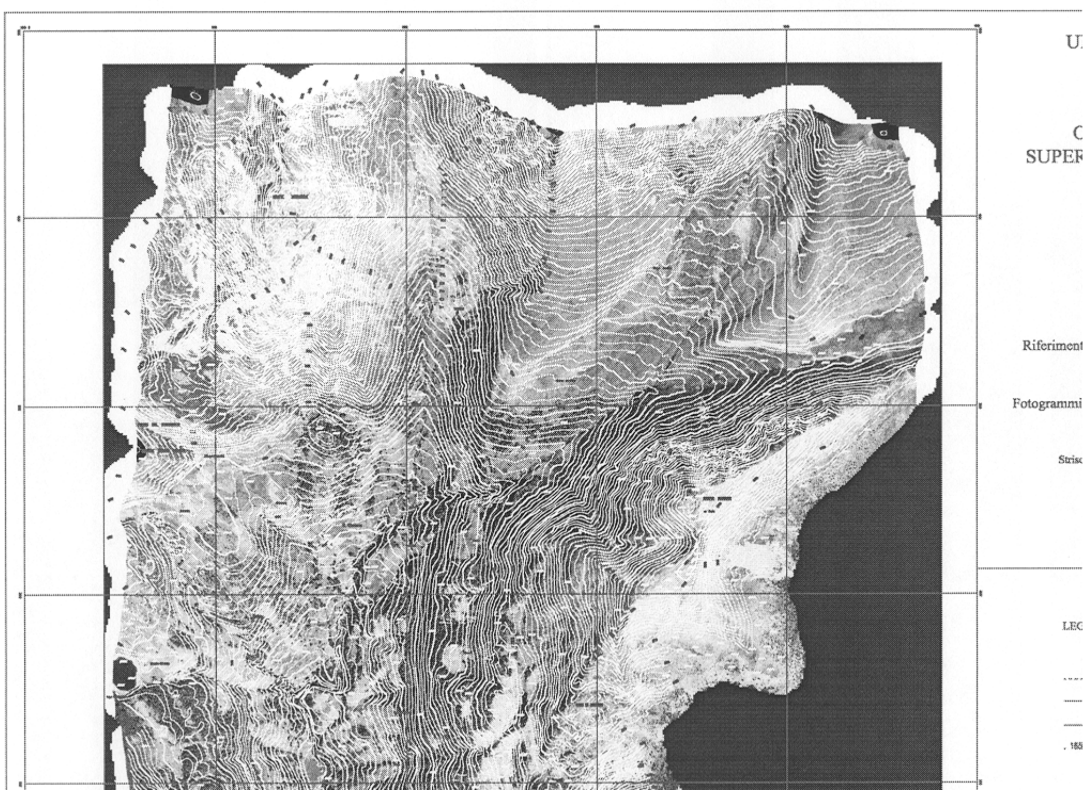
Updated technical map of Mortirolo (BS) by shading

Original aerial image of Mortirolo (BS)



Orthoimage of Mortirolo (BS)



Orthoimage of Mortirolo (BS) with superimposition of an updated topographt by contour lines

PART III – A CERTAIN TYPE OF NUMBERS WHICH REPRESENT SPACE AND TIME AT DIFFERENT DIMENSIONS

Introduction

The impressive spread of technology produced a great progress in the field of Geo-Information and Geomatics, with the hope of ulterior improvements. Unfortunately, a limit to such implements is posed by an objective difficulty to manage (and to understand) available data and to build models from them.

The object paradigm supplies a high level of abstraction too; it is able to model the system according to the definition of data (and objects), as well as to operate on the concept of attributes and relations, being both stored as an integrating part of the same objects.

Object oriented representation permits to create complex objects, like polygons, 3D bodies, etc., analyze and manipulate them like single objects (even if they are combinations of objects). This characteristic eliminates the necessity of clarifying all geographic and semantic attributes of the objects. Furthermore object-oriented approach allows for easily description of new types of data, defining operations on new objects and structuring the objects in a hierarchical way

Therefore object, eventually dynamics, modeling is a technologically advanced instrument, which increases the capabilities of Informative Systems. Unfortunately the use of this instrument is still rare; in fact it remarkably increases the complexity of realization and management of the informative system. Indeed for example, several applications in this field require the knowledge of 3D geometry and topology: as well known, operating with 3D space is considerably more complex than working with 2D models.

Particularly for management of spatially referenced data it is necessary to create new entities (i.e. polygons, 3D bodies, etc.) and to define rules for their processing, like distance, incidence, adjacency, etc. However in object oriented systems, some problems of difficult solution, presented by relational models, are still not completely solved. The development of these systems is going in two different directions: adding all typical functions of the object paradigm to a relational databank, or constructing a new independent system. The first approach is very robust, because the databanks are a mature product, while the second approach is still an open issue in the scientific and technological research.

Consequentially object modeling allows the widest modalities to represent 3D bodies and figures for their studies and analysis. The relations between characteristic elements (i.e. points, lines, surfaces and 3D bodies) determine the validity of this modeling and their sets have correspondences with the groups of symmetry in the one-dimensional, two-dimensional and three-dimensional spaces of the formal algebra. It is remarkable that many considerations able to allow quick choices, avoiding the Boolean computation of all possible combinations – out of which the ones geometrically possible should be taken – have a wide usage of the numbers 7, 17 and 32.

As we look through other fields of human knowledge, we often find strange properties of numbers; specially of the over-said numbers, in far distance fields: going from Psychology to figurative arts, from Linguistics to religious thought. Although we should never forget that any numbering system is somewhat arbitrary, still fancy and rigor are both necessary for a sound evaluation of the world.

Relations among the object elements

In order to construct a formal data structure, it is necessary to identify geometrical features of objects and their relations.

As the first step, one may operate with different methods, any case, once complex objects have been someway defined, it is necessary to establish their mutual relations in the space, where they are located (in terms like: at the right or at the left, inside or outside, over or under, intersects, etc.). Thus we can notice that the topological/geometrical relations in one dimension are 7; they are however 17 in 2D environment and 32 in 3D space. Indeed, in order to obtain a broad starting point, very general relations among complex objects are defined, as follows:

- 3 classes of relations in the one-dimensional space: between points, points and lines, lines;
- 6 classes of relations in the two-dimensional space: beyond to those of the previous case, between points and surfaces, lines and surfaces, surfaces;
- 10 classes of relations in the three-dimensional space: beyond to those of the previous cases, between points and 3D bodies, lines and 3D bodies, surfaces and 3D bodies, 3D bodies.

In one dimension, as explained in the GIS chapter, topological relations are 7:

- Point – point: separate
  coincident
- Point – line: external
  internal
- Line – line: separate
  connected
  internal.

In two dimensions, 10 new relations join to the previous ones, defined for the one-dimensional case, reaching the number of 17 topological relations:

- Point – surface: external
  internal
- Line – surface: external
  connected
  secant
  internal
  enucleating
- Surface – surface: external
  connected
  internal.

Finally in three dimensions, 15 new relations join the previous ones, defined for the two dimensional case, reaching the number of 32 topological relations:

- Point – body: external

internal

- Line – body: external
  connected
  secant
  internal
  enucleating
- Surface – body: external
  connected
  secant
  internal
  enucleating
- Body – body: external
  connected
  internal.

The main geometrical relations among primary elements are not lesser than:

- 10 geometrical relations, for the one-dimensional case;
- 32 geometrical relations, for the two-dimensional case;
- 230 geometrical relations, for the three-dimensional case

Especially, in the three-dimensional case, it is quite easy to reach the number of 230 elements. In fact, while a point is a point and a body a body, a line can be open or close, and it can occupy an open area or close area, as well as a volume; moreover a surface can be open or close, and it can occupy a volume.

About Symmetries

Many examples of symmetries due to rotation, translation, reflection can be noticed both in arts and sciences. Another type of transformation – not geometrical however – is permutation, starting with Latin squares as studied by Euler.

A permutation symmetry may also take place in abstract cases: the proposition "X is a relative to Y" is symmetrical: actually, the meaning won't change if Y takes the place of X. This does not hold, when the proposition is "Y is the son of X" as for as the logical reasoning is concerned.

According to the psychoanalyst Ignacio Matte Blanco, logic of unconscious system is different from one valid for the "normal" one: "… the unconscious system deals with the inverse relation the same as if it were identical to the direct" too. In other terms, asymmetric propositions become symmetrical: in the above quoted use, Y is the son of X, of course in the world of our dreams, that is the unconscious (or changing a word "Y is the father of X").

We can then place problem of symmetry at the crossroads of science, art, psychology. The Gesthalt Psychology has widely enlightened the relevance of symmetry: as different organizations are theoretically possible of information available to our senses, human mind choices the simple one and the symmetrical one.

A proper language for symmetries is the group theory: the whole of all the transformations gives place to a group. Abel and Galois, two very young people, first found group theory: even their lives, in a certain sense, appear as symmetrical. Indeed they both started with the same problem (resolution of fifth grade equation) to reach the general theory of groups, in two different ways [3].

The prohibition of figurative arts for Hebrews and Arabs carried to the development of a pure abstract and geometrical art and to the exploration of possible types of decoration. In this field, the most elevated result was reached in Granada in the 14th century, with the tessellation of the Alhambra.

Although the number of decorations is nearly unlimited, they are limited as far as the type of symmetries adopted for their repetition. From a mathematical point of view, these symmetries can be classified based on possible transformations, which leave them unvaried: translation along a direction, reflection with respect to a straight-line and rotation around to a point.

In 1891, Fedorov demonstrated that only 7 types of symmetry for liner decorations and only 17 for the planar ones exist. Furthermore the planar groups can only present rotational symmetry for angles of 180°, 120°, 90°, 60°, which correspond to the axial, triangular, squared and hexagonal rotation angles.

If most common examples of linear and planar symmetry are decorations, most common examples of spatial symmetry are crystals.

From 1849 with Auguste Bravais, the crystallography has been one of first fields of application of group theory of symmetries. In 1890, before demonstrating the analogous results for the groups of planar symmetry, Fedorov had demonstrated that only 230 types of spatial symmetry exist.

The first part of the 18th problem of Hilbert asked whether groups of symmetry in n dimensions are a finite number for every n. In 1910, a positive answer was given by Ludwig Bieberbach; however an explicit relation is not yet found giving the number of groups of symmetry for any n given. In fact, the existence of 4783 types of four-dimensional symmetry was demonstrated only in ’70.

Symmetry means invariant respect to a group of transformations and, if the transformation is a distance, the symmetry becomes an isometry. In the plan, four types of isometry exist: translation, rotation, reflection and glisso-reflection [4].

The study of topological and geometrical relations among primary elements and groups of symmetries, in the spaces where the complex objects are located, shows particularly curious identities between the number of these relations and the cardinality of groups of symmetries.

In fact, as above shown, topological relations in one dimension are 7, as many as are the elements of the group of liner symmetries, which have one single direction of translation.

The same analogy is evident in two dimensions where, in correspondence to 17 topological relations, an identical number of elements forms the group of symmetries in the plan, considering two directions of translation.

Still to the 32 topological relations, characterized in three dimensions, correspond the elements of the group of symmetries in 3D space, considering three directions of translation and crystallographic restriction. Furthermore considering the main geometrical relations:

---

[3] Also both of them met great difficulties with the academic world, had lots of troubles in their private life and died very young. Moreover they should special interest for social life, for art and politics. Singular was also their death: Abel died with malnutrition and related diseases, Galois lost his life in a duel, for a woman.

[4] Notice that rotation can be replaced by two suitable reflections, as well as translation; moreover glisso-reflection can be reduced to one reflection, followed by one translation.

- 10 (number of elements in one-dimensional case) corresponds to the number of elements of the group of symmetries in plan, considering crystallographic restriction;
- 32 (number of elements in two-dimensional case) corresponds to the number of elements of the group of symmetries in 3D space, again considering crystallographic restriction;
- 230 (number of elements in three-dimensional case) corresponds to the number of elements of the group of symmetries in 3D space, without any restriction,

being 4 the number of elements of the group of liner symmetries, considering crystallographic restriction.

Numbers running after each other

It is remarkable that many considerations able to allow quick choices, avoiding the Boolean computation of all possible combinations-out of which the ones geometrically possible should be taken – have a wide usage of numbers 7, 17 and 32. We can also add to the over said consideration about the groups of symmetries, that we have 7 Euclidean and spherical finite 2D surfaces (plane, cylinder, Moebius strip, torus, Klein bottle, sphere, projective plane) and 17 3D Euclidean hyper-surfaces (aside from the 3D Euclidean space and the Euclidean space with a hole, i.e. a torus in a translation motion). On the opposite, hyperbolic 2D surfaces, as well as hyperbolic and spherical 3D hyper-surfaces are both infinite: by now, studies about 4D surfaces of any type are not yet completed.

First of all, we have 5 regular polyhedrons, (known as Platonic, however Pythagoric), in addition to (13+4) semi-regular and concave polyhedrons (Archimedean and due to Keplero-Poinsot): the total is 17, as below:

Regular 3D bodies	Semi-regular polyhedrons	Concave bodies by Kepler – Poinsot
1. tetrahedron	1. cube – octahedron	14. small star-dodecahedron
2. octahedron	2. icosa – dodecahedron	15. big star-dodecahedron
3. cube	3. truncated tetrahedron	
4. dodecahedron	4. truncated cube	16. big dodecahedron
5. icosahedron	5. truncated octahedron	17. big icosahedron
	6. truncated dodecahedron	
	7. truncated icosahedron	
	8. rhombi cubic octahedron	
	9. truncated cube – octahedron	
	10. rhombi icosa dodecahedron	
	11. truncated icosa – dodecahedron	
	12. snub cube	
	13. snub dodecahedron	

Besides in the last times, a group of no lesser than 92 convex solids has been defined, proven as complete by Zaigaller, the Johnson's solids, having all faces made with the regular polygons, without being any of over listed polyhedrons, nor prisms or anti-prisms. Other than pyramids, domes and rotund (simple and modified), and augmented prisms, they have 7 Platonic modified solids, 19 Archimedeon modified polyhedrons and some mixed solids (3 of them are called sphenoid-corona-simplex, augmented and mega). A super class of 32 elements is formed with for the last but one class, the selected members of the last class, plus three more complex concave polyhedrons and 7 truncated concave polyhedrons (obtained from the last ones and from previous four concave polyhedrons by Kepler and Poinsot).

Then quite curious is the fact that the number 2 has a logical link with opposite couples, first of all the right-left one, and also that, after this number, the other ones are far from each other by multiples of 5 (the fingers of a hand):

$$2 + 5 = 7 \qquad 7 + 10 = 17 \qquad 17 + 15 = 32$$

Also remember that 32+20=52, i.e. the number of weeks in a solar year of the earth: that adds a timely dimension – a fourth one – to the three dimensions of a 3D space. For the last: 52 + 25 = 77, 77 = 7 $\times$ 11, and 11 is the first prime number after 7. The numbers running each after other never come by chance, so we should pay attention to the following considerations.

Hence grammatical and syntactic rules supply interesting additional examples, such as the qualitative degree of plurals:

- a few, some, several, many,
- less than, as many as, greater than,

in an evolved specific character of human language and offer a number of possibilities equal to 7, like everything else that may be referred to a linear order. Moreover adverbs and adverbial circumlocutions for time and locations:

Now	Here
Before, after	Near, far
Often. Rarely	Forward, backward
In short, at long	Rights, lefts
	Above, under
	Inside, outside
	Nearby, in touch
	Along, through
	Densely, sparsely

are grouped in order to represent and explain phenomena and processes time-variant, i.e., dynamic, and geo-referential (mainly with two-dimensional spatial reference or in 2 ½ D). It should then be remarked that 7 and 17 are recurrent numbers in linear and planar context, for instance in symmetries of friezes and mosaics, topological relations 1D and 2D.

Hereby follows the table of locative case, numbering nine:

Locative cases:	State in	Motion from	Motion to
interiority	inexiv	elativ	illativ
superficiality	superexiv	delativ	sublativ
adherence	adexiv	ablativ	allativ

It is the maximum number, for some Hindu-European languages: however only for them a complete classification and comparative studies are available. One may also get a number of 16, when adding the "motion to" circumscribed, (when it is never referred to an own specific case), also including exteriority among positions, or even 32 if one takes into account compound objects, in addition to simple ones. The same number is recurrent in spatial analysis too, for instance in crystal symmetry, or in 3D topological relations.

Theory of numbers

Some curiosity has been widely known for many centuries: for instance, correspondence between the order of 30- and 28-day months alternated to the ones with 31 days (5+7=12, from the month following spring equinox) and the harmonic series of notes (see also the 17th century studies on the well-tempered harpsichord:

N	Months	Days	Musical notes	Tastes
1	April	30	La flat	Black
2	May	31	La	White
3	June	30	Si flat	Black
4	July	31	Si	White
5	August	31	Do	White
6	September	30	Re flat	Black
7	October	31	Re	White
8	November	30	Mi flat	Black
9	December	31	Mi	White
10	January	31	Fa	White
11	February	28 or 29	Sol flat	Black
12	March	31	Sol	White

In his time, a young and poor Indian mathematician, Ramanujan (1887-1920) has utterly extended the field of such add properties by remarking, e.g:

$$1729 = 1000 + 729 = 10^3 + 9^3 = 1728 + 1 = 12^3 + 1^3$$

$$1089 \times 9 = 33^2 \times 3^2 = 9801 \qquad \text{(reverse of 1089)}$$

$$2178 \times 4 = (2 \times 1089) \times 2^2 = 8712 \qquad \text{(reverse of 2172)}$$

which means that no smaller number can be expressed as a double sum of two different cubes also no other number lower than 10.000 can be decomposed into two factors, one of them being itself written as reverse.

The Theory of numbers was first due to Pythagoras, Euclid and Eratosthenes, had some authors in the Middle Age (Fibonacci) and a new flourishing between ‘600 and ‘700, first with Marsenne and Fermat, later with Euler and Legendre. A new approach in the XIX century, after Gauss, has come from many mathematicians: it has proved to be a really fascinating and surprising game (cubic set of four numbers are more than Pythagorean terns, as linked to the so called last theorem by Fermat, only recently demonstrated: $a^n + b^n = c^n$, being $n \leq 2$ and a, b, c integers).
It is still to demonstrate, among other things, the reason why perfect numbers (i.e. numbers equal to the sum of all their divisors, included one and except the number itself: 6, 28, 946, 8128, …; about fifty are known at present) are all even. Merely for a curiosity, and for a natural sympathy with the name two numbers are called friendly, when they exchange the sum of each other’s divisors (e.g., 220 and 284, 1184 and 1210, 2620 and 2984, 5020 and 5564, 6232 and 6368, …), for them, the first odd couple is 12285 and 14595, while a mixed couple, theoretically possible, is not yet known, for numbers below one million.
Moreover a number of psycho-linguistic determination show, possibly independently from each other (anyway the correspondence is surprising), that the above said numbers 7 (with an interval 5-9), 17, 32 are:

- maximum amount of information that can be stored in short term memory;
- maximum amount of that can be easily stored in long-term memory (by re-iteration);
- maximum amount of information that can be stored in long-term memory by so-called method of loci,

This method (loci), already widespread in the Greek-roman world for rhetorical applications and re-used in the Renaissance and later, as an art of memory technique, realizes an association of a selected arguments with more familiar information, easier to remember.

Conclusion

Everybody know the Eleatic paradox stating that the quick-foot Achilles shall never recovery on the tortoise: other paradoxes of the same origin cope with concepts of zero and infinity. Nowadays, zero is a number, quite well known, and infinity is the limit of an impossible division between a finite numerator and a null denominator. However zero comes to Europe in the Middle Ages, imported by Arabs, which learned it from Persians, in turn taking the concept from Indians, maybe the only ancient civilization able to make up an Arithmetic devoted to calculus.
The lack of number zero is the cause of infinity being thought of by paradoxes. Although the entire western antiquity (Middle East, North Africa, Mediterranean Europe) had a clear concept about void, absence and nothing, the number zero could not be conceived. So, the first century flows from the year one to the hundredth, as (even today) the ordinal zero does not exist. Notwithstanding the number zero has come to Europe in the Middle Age, the concept of limit has been formed only between the seventeenth and the early nineteenth century, by Leibnitz, Newton, Euler, Lagrange, Lapalce, Gauss and Cauchy: as a natural consequence of the discovery of infinity comes a rigorous definition – in the mathematical sense – also of the concept of infinity (stressed by Weierstrass, Dedekin, Cantor, Peano and Hilbert). Sure it is not a number, but a mathematical devices, of a great use to face a lot of simple or complex studies.
A laic behavior means thinking and writing about religious themes, in full respect for people who have them as important of their life, but asking for a correspondent respect for people having little or no interest for such

problems. Any other behavior is obviously fundamentalist and/or obscurantist. In this view, we take into consideration some numbers from Western religions (Hebraism, Christendom, Islam), compared with other numbers from Oriental-type religions (Hinduism, Buddhism, Confucianism, Taoism, Shintoism). The number 7 is the number of the days of Creation in the Book of Genesis in Bible: from it comes evidently the number of days of our week. Also the Decalogue lists 10 commandments, divided into three (a complementary number between 2 and 5) plus 7 (Exodus book). The same numbers are quite frequent in other sections of the same text.

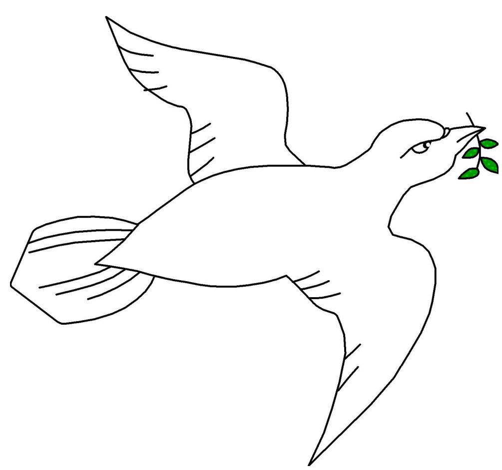
However remarkable, is the number 5 of Islamic prescriptions, as reported in the Koran. Even without a deep knowledge of oriental religious thought, it is noteworthy to quote the 4 lifecycles, the 4 aims of human life, the 4 classes of Hindu society, as well as the 4 noble truths and the noble 8-fold pathway of Buddhism (to which are also linked some ancient Chinese and Japanese religious schools). They put in evidence different numeric games: actually, 4 and 8 are the first two powers of 2; also they do not consider the number 5, nor 3 or 7 (difference and sum, respectively).

In the history of mankind, agricultural and domestication of animals has been widespread in many countries, from China to Mediterranean basin. The history of human cultures show since from Antiquity an evident diversification between East and West: it should be the task and the good wish of our age to overcome this ancient dichotomy.

A plain conclusion of any reflection, serious or amusing, should always be an open one. Numbers 7, 17 and 32 are surely meaningful: 7 is to be found in may tales from Romanticism, 17 in Tzigane stories, 32 is in the Jewish Cabala.

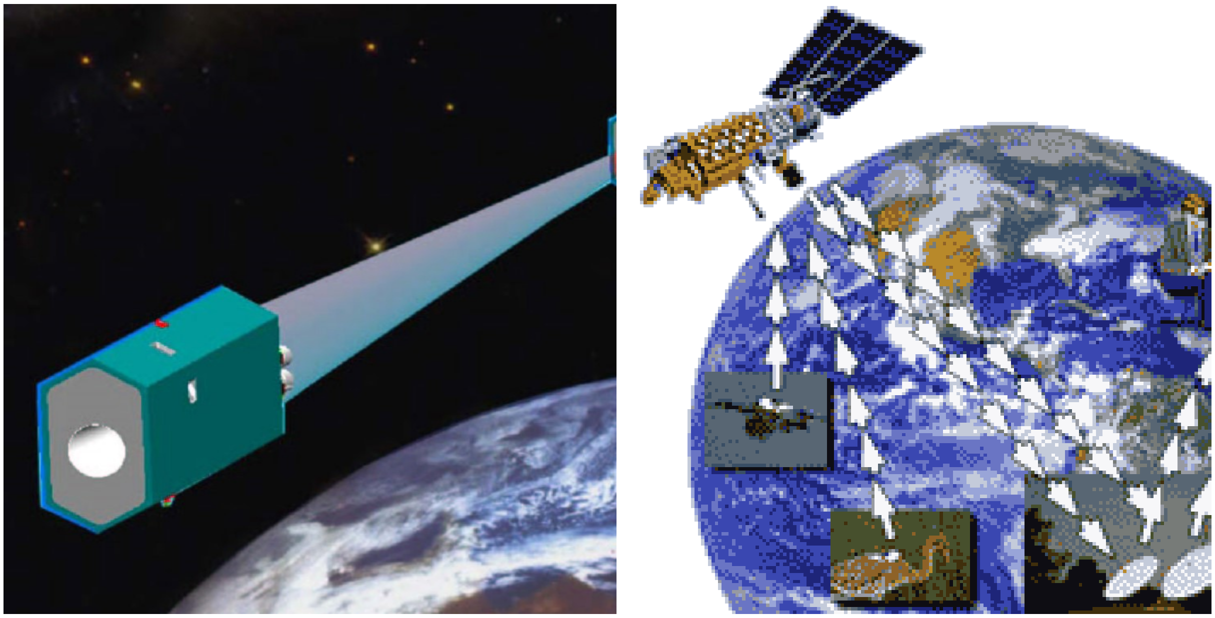
In any case, any numbering system is arbitrary in itself: it is only a useful device, able to check and understand real world, in its complexity and variability. So an easy quest could find other tales based upon prime numbers 13, 19 and 23 (maybe, as related to the 24 daily hours), and also upon the following prime numbers 29 and 31 (maybe, as related to the days of longer months). However, other tales are related to the number 91 (not a prime one), 93 and 97, which are prime, but comparatively large.

Although we should never forget the arbitrary of numbering, still fancy and rigor are both necessary for a sound evaluation of the world. Finally the present considerations want only to express a qualitative point of view, while mathematical implications would often require to proof complex theorems.



Pablo Picasso (Paris, 1949): Peace pigeon,
drawn for the Congress of World Peace

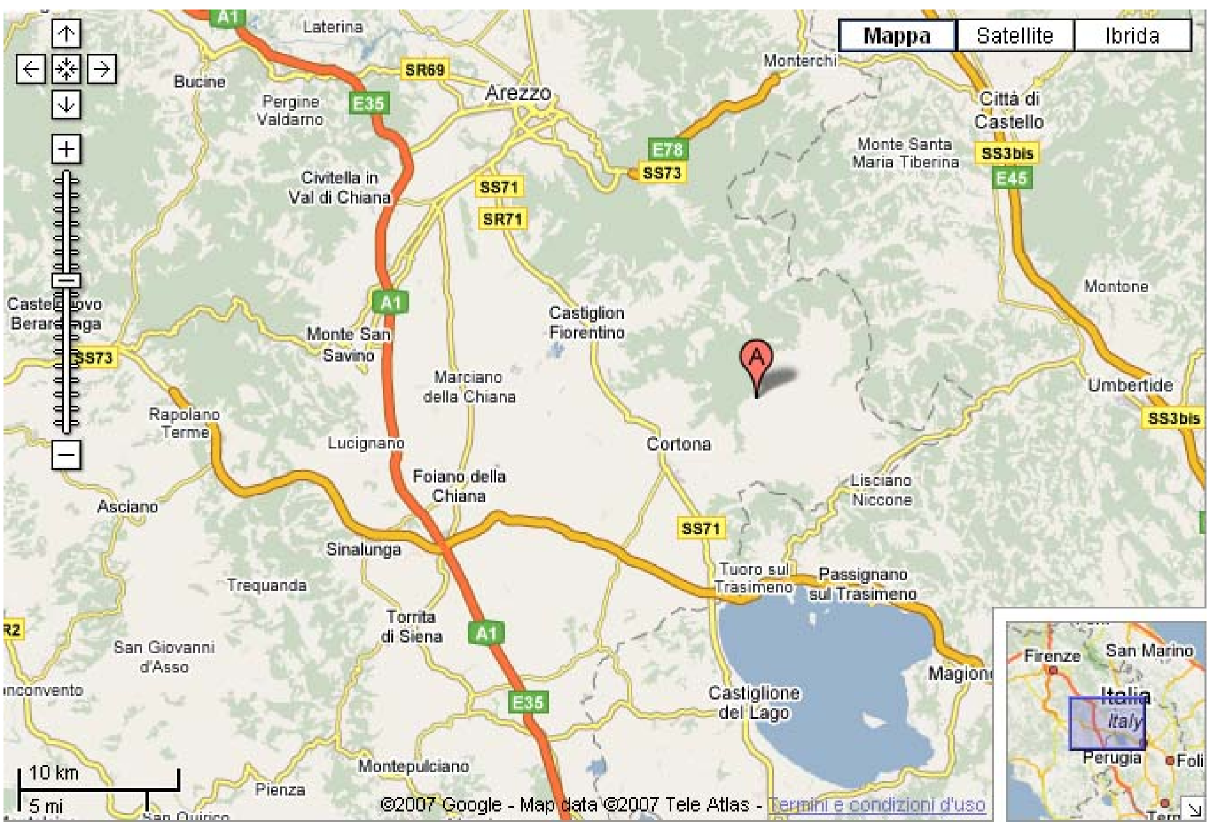
Appendix E – Examples of modern survey and mapping data sources



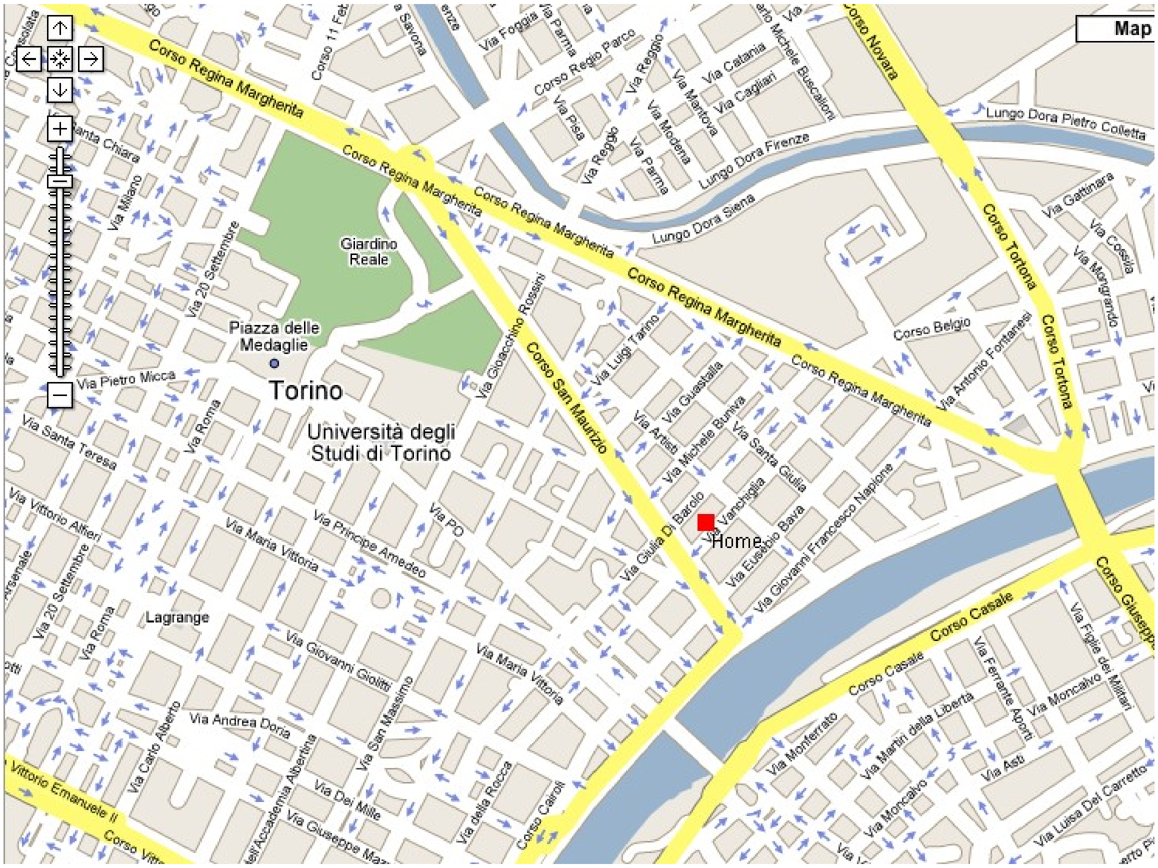
Satellite to satellite tracking and satellite geodesy (e.g. GPS)

Google earth

Google Maps (at small scale)



Google Maps (at medium scale)



Google Maps (at large scale)

Video-surveillance (e.g. by CCD camera)

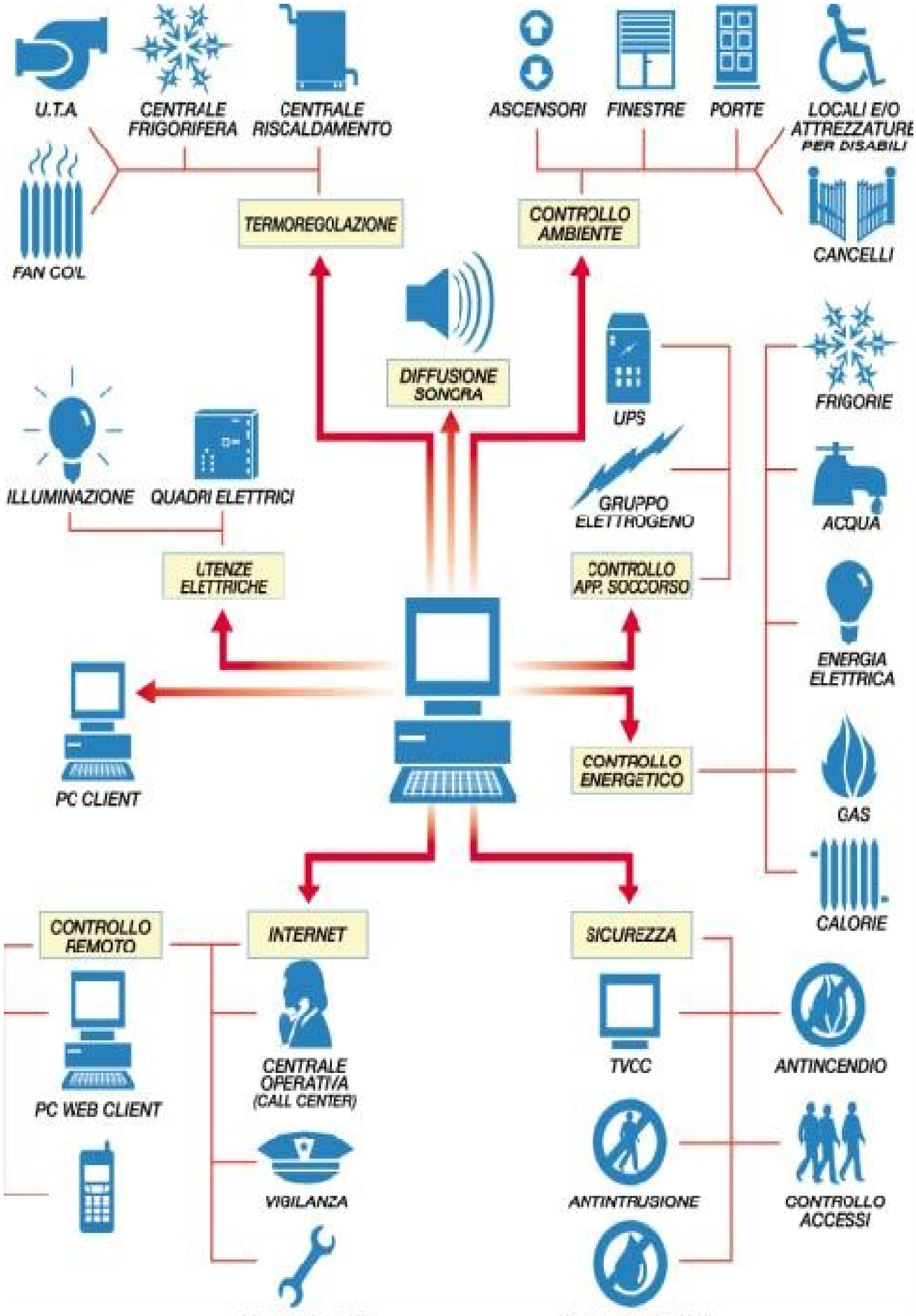
GPS antenna

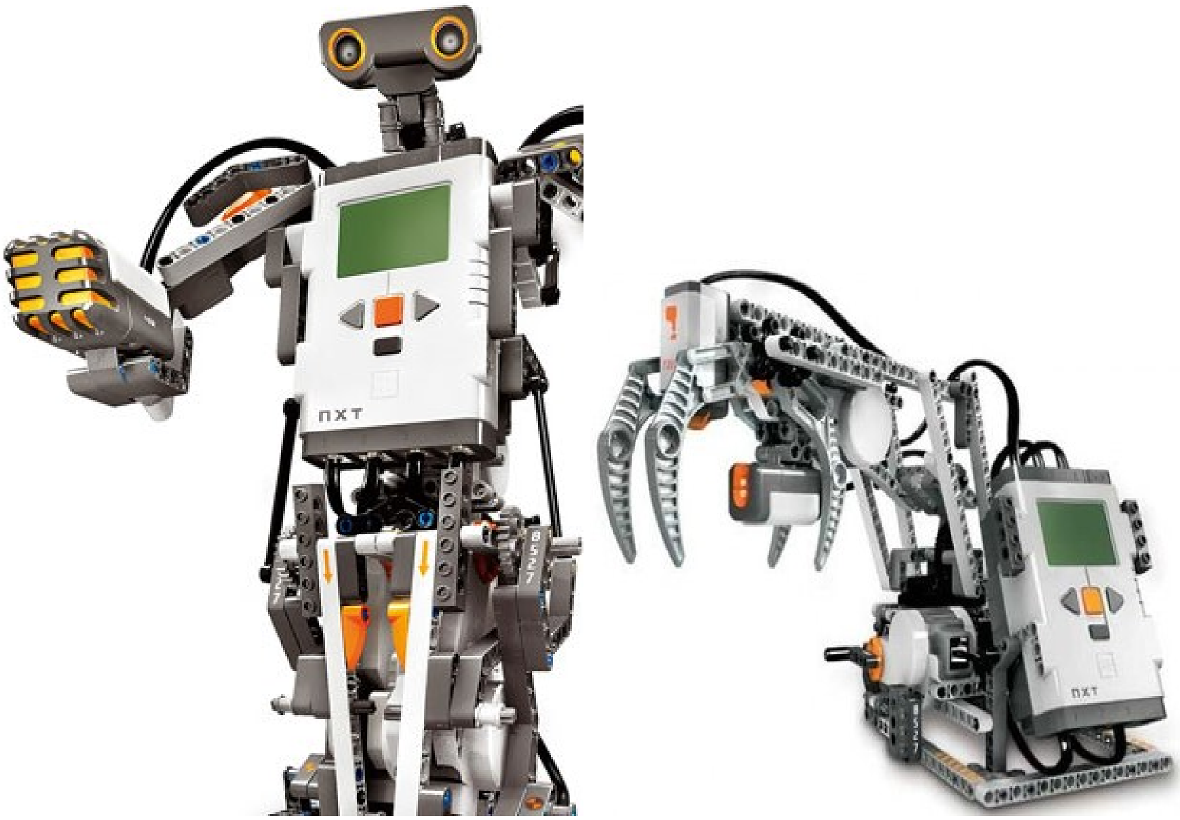
Drone

Digital camera

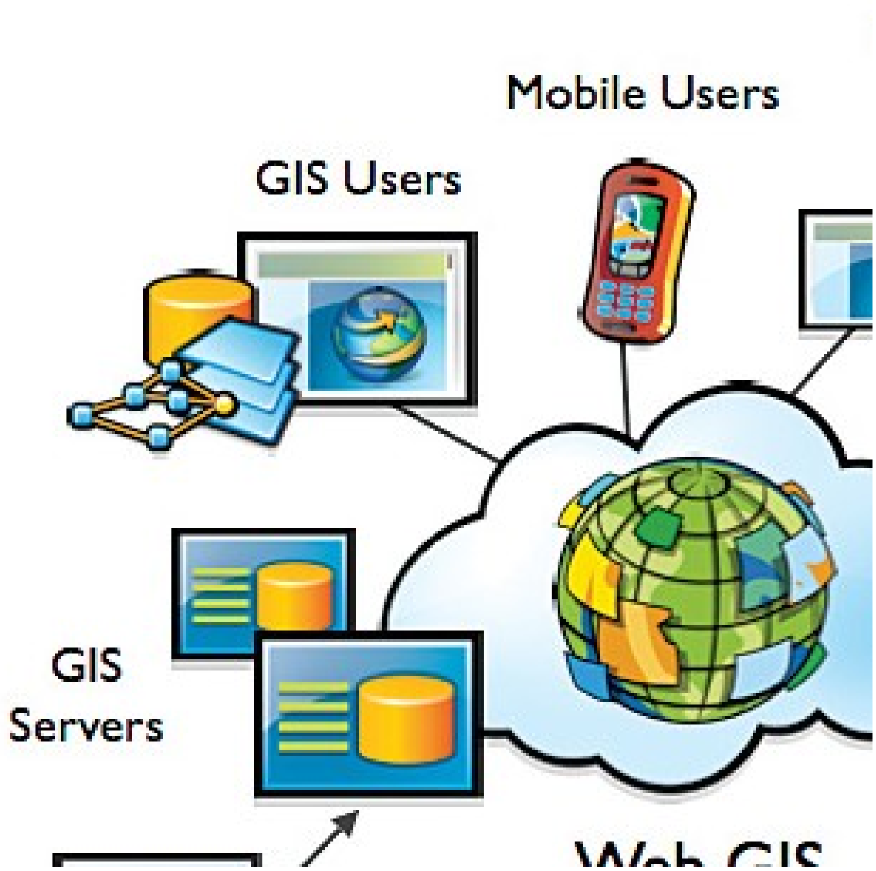
Digital theodolite

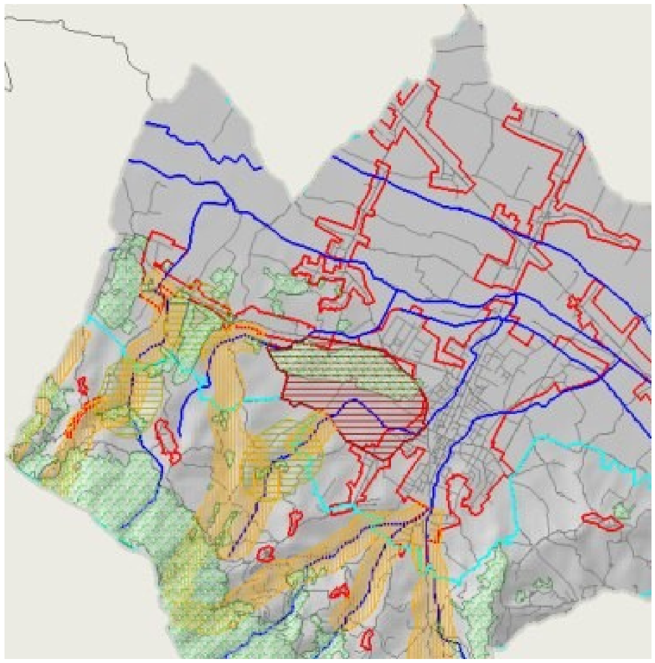
Digital level

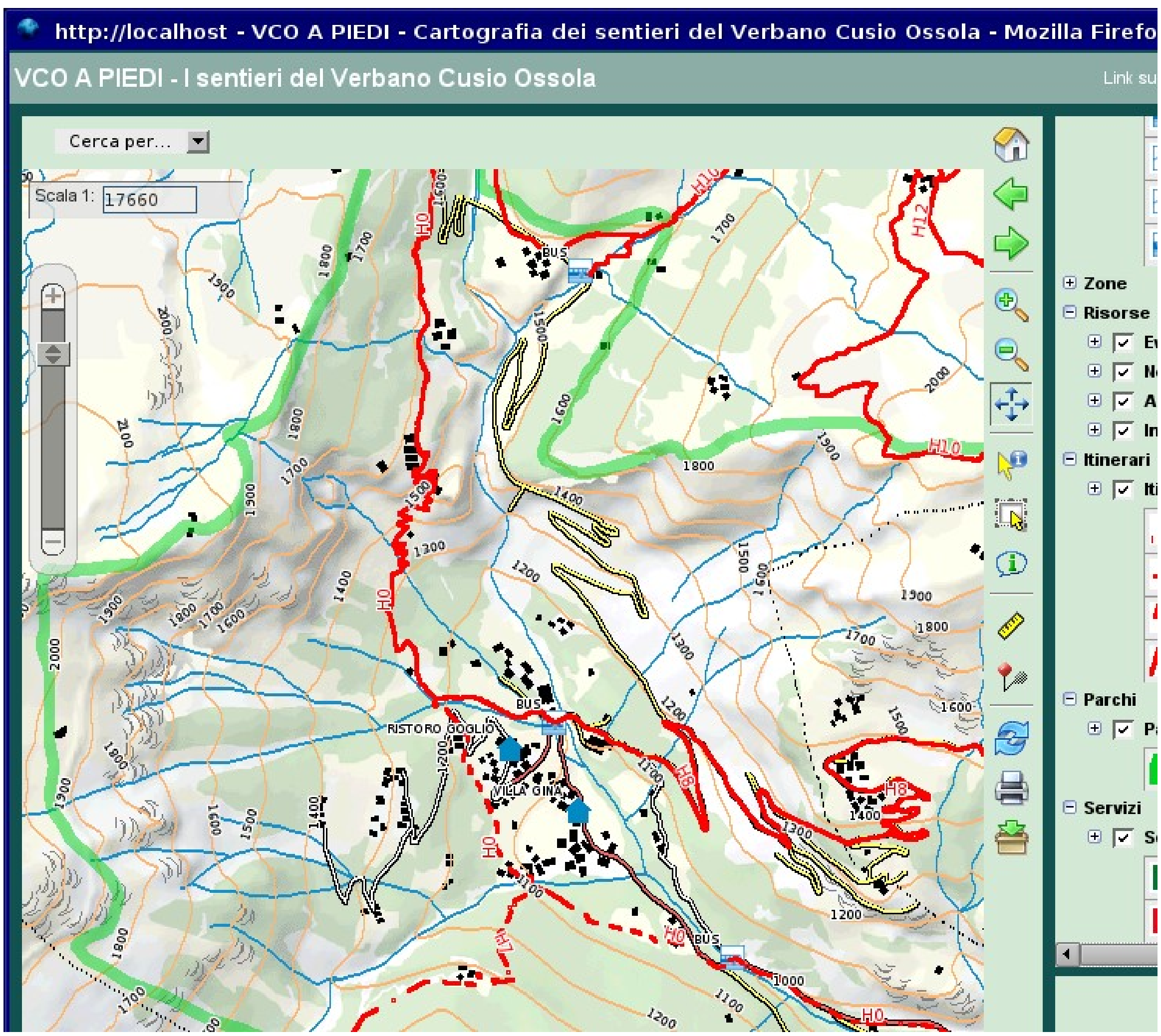

U.T.A.
CENTRALE FRIGORIFERA
CENTRALE RISCALDAMENTO
ASCENSORI
FINESTRE
PORTE
LOCALI E/O ATTREZZATURE PER DISABILI
FAN COIL
TERMOREGOLAZIONE
CONTROLLO AMBIENTE
CANCELLI
DIFFUSIONE SONORA
UPS
FRIGORIE
ILLUMINAZIONE
QUADRI ELETTRICI
GRUPPO ELETTROGENO
ACQUA
UTENZE ELETTRICHE
CONTROLLO APP. SOCCORSO
ENERGIA ELETTRICA
PC CLIENT
CONTROLLO ENERGETICO
GAS
CALORIE
CONTROLLO REMOTO
INTERNET
SICUREZZA
PC WEB CLIENT
CENTRALE OPERATIVA (CALL CENTER)
TVCC
ANTINCENDIO
VIGILANZA
ANTINTRUSIONE
CONTROLLO ACCESSI



Robotics and home automation



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VILLA GINA
BUS
H0
H8
H10
H12

Web GIS

Appendix F – Comparison of measure units and data processing hardware

SCS measures	**SCS equivalent**	**SI equivalent**
mil	1/25 *line*	0,0254 mm
line	1/4 *inch* or 25 *lines*	0,635 mm
inch	1/4 *hand* or 4 *lines*	25,4 mm
hand	1/3 *foot* or 0,4 *span* or 4 *inches*	101,6 mm
span	1/2 *cubit* or 2,25 *hands*	228,6 mm
foot	1/3 *yard* or 3 *palms*	304,8 mm
cubit	1/2 *yards* or 2 *spans*	0,4572 m
yard	1/2 *fathom* or 2 *cubits* or 3 *feet*	0,9144 m
fathom	0,36 *perch* or 2 *yards*	1,8288 m
rod, *pole* or *perch*	1/4 *chain* or 2,75 *fathoms*	5,0292 m
chain	1/10 *furlong* or 4 *perches*	20,1168 m
furlong	1/8 *statute mile* or 10 *chains*	201,168 m
statute mile	8 *furlongs*	1609,344 m

Measures of length

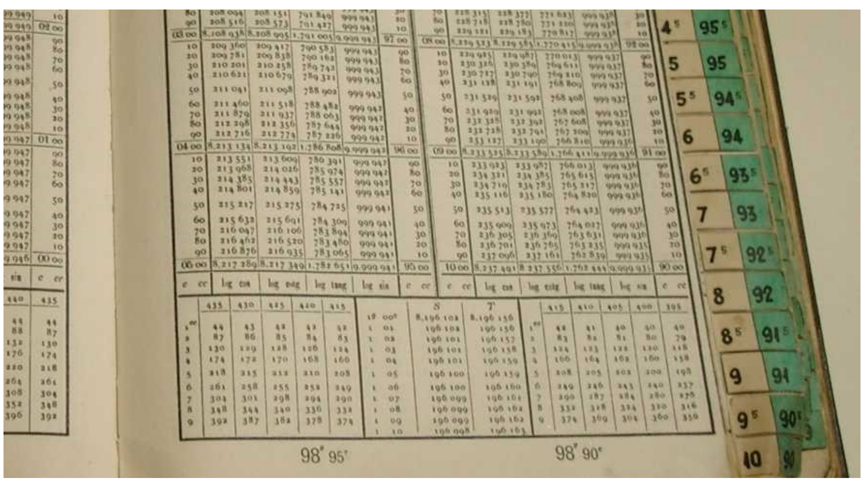
SCS measures	**SCS equivalent**	**SI equivalent**
liquid ounce	1/5 *gill*	28,4 ml
gill	5 *liquid ounces* or 1/4 *pint*	142 ml
pint	4 *gill* or 1/2 *quart*	0,568 l
quart	1/4 *gallon* or 2 *pints*	1,1364 l
gallon	4 *quarts*	4,546 l

Measures of capacity

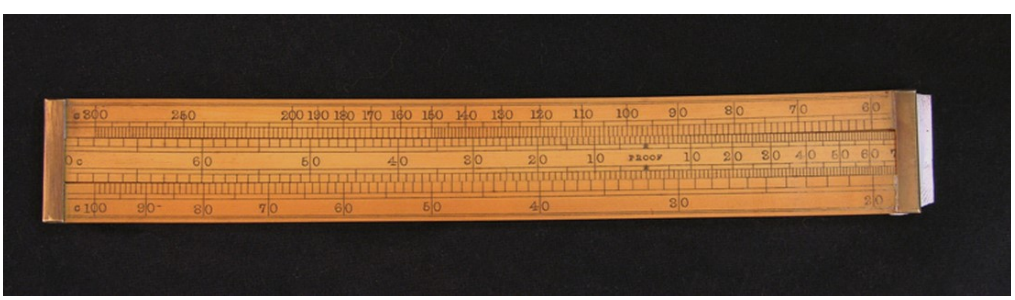
SCS measures	**SCS equivalent**	**SI equivalent**
grain	1/7000 *pound*	~64,798 mg
dram	1/16 *ounce*	~1,771845 g
ounce	1/16 *pound* or 16 *drams*	~28,349523 g
pound	1/14 di *stone* or 16 *ounces*	453,59237 g
stone	1/2 di *quarter* or 14 *pounds*	~6,35 kg
quarter	1/4 di *hundredweight* or 2 *stones*	~12,7 kg
hundredweight	1/20 di *long ton* or 4 *quarters*	~50,8 kg
long ton	20 *hundredweights*	1016,0 kg

Measures of weight

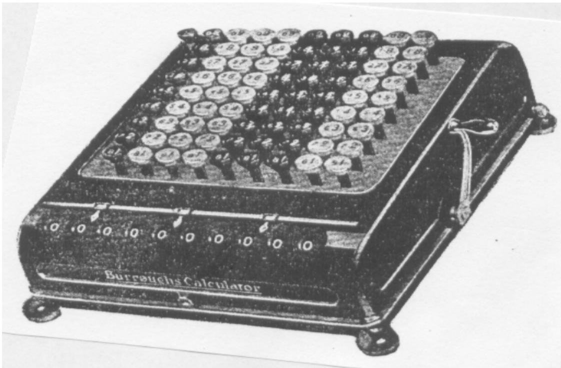
Data processing hardware

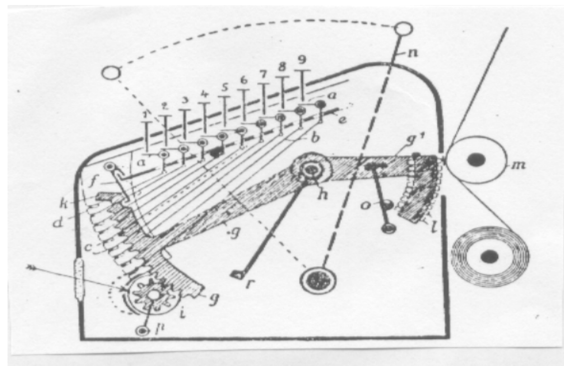


Logarithmic and trigonometric tables (18th century)

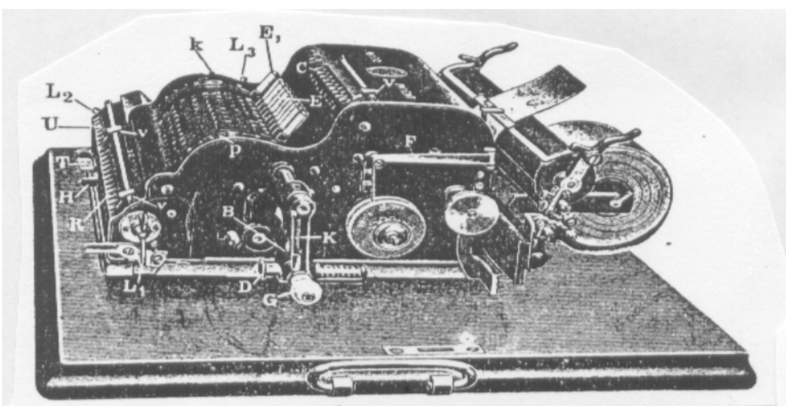


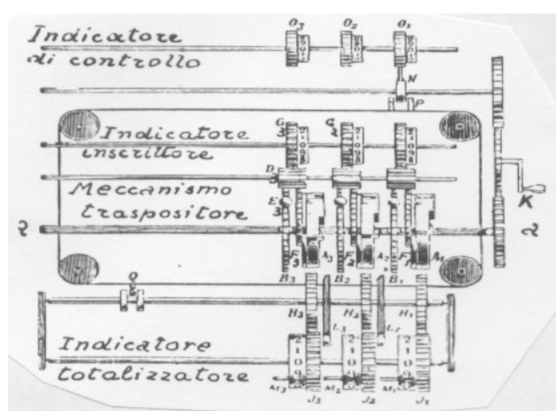
Slide rule (19th century)



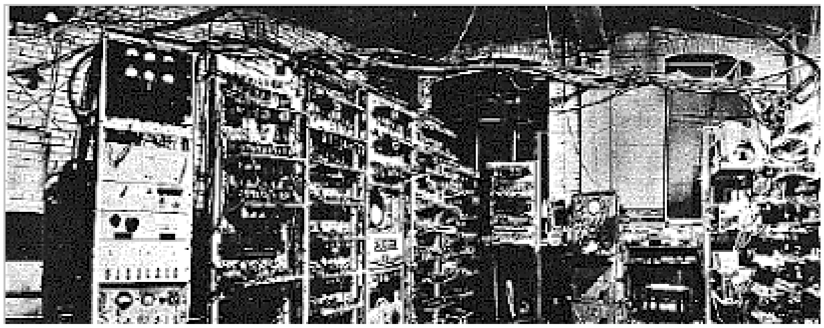


Burroughs addition and subtraction machine (18th century)

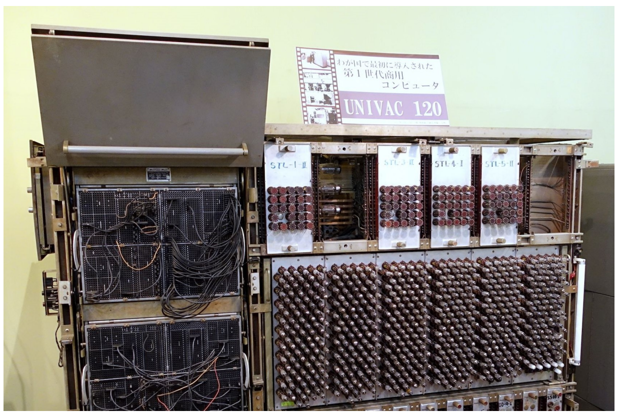




Brunswig four operation machine (19th century)



First computer in the ‘50s of 20th centuty



Second generation computer in the ‘70s of 20th century

Workstation in the ’90 of 20th century

PC in the ‘10th of 21 century

Acknowledgement

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