DATA PROCESSING AND GEOMATICS MEET HUMAN SCIENCES

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Abstract – A mixed collection of advanced mathematical and statistical methods, together with a summary of some ideas, coming from the world of human sciences, is presented with the aim to join the two cultures.

FOREWORD [1]

Humanism must help us: not because it's the other half of thought (i.e. it contains questions and solutions) and not because it represents the other point of view, but because it holds together different points of view and it's able to explain them.

Times explain technology, but humanism explains times.

- Technological knowledge captures the *novum* of the present. It has his gaze turned forward and adopts the replacement paradigm of forgetfulness, chasing the urgency of the *ars respondendi*. It lives in space, is familiar with life understood as "vital principle" and simplifies complexity.
- Humanistic knowledge knows the *notum* of history. It looks back and forth (the *simul ante retroque prospiciens* by Francesco Petrarca [2]), adopts the cumulative memory paradigm and knows the urgency of the *ars interrogandi*. It lives in time, is familiar with life understood as "purely individual existence" and interprets complexity.

  "Man goes through the Oceans – so the first chorus of the old Greek tragedy Antigone (by Sofocle) recites ideally listing the benefits of Prometheus – he tames the fairs, learns by himself words and the thought of wind and desire of politics and he has the knowledge of the technique, beyond every hope. He devised every remedy for irremediable ills, but he found no remedy at death. This is why it is the most wonderful and tremendous being". He solves problems, but he remains a problem to itself.

- The machine is ordered, predictable and programmable;
- vice-versa man is disordered, unpredictable and imperfect.

We can create the perfect shape of the circle, but we remain – as said the Pythagorean philosopher and physician Alcmeone (in 6th century BC) – an unfinished circle, an arch, "the arc of life". This is why we die: beginning and end coincide in a perfect circle and don't coincide in an arc. Death and life are inscribed in the word "bios", for a wonderful and tremendous linguistic ambiguity. When we speak of men, it's urgent to resort to a sort of deviation from the machine, of waste: between finitude and utopia, and of a *clinamen* of Lucrezian memory. This is our condition but also our strength: technique is not enough and politics is not enough. The truth is that while scientific-technological knowledge runs fast and celebrates its triumphs, regardless of every *telos* and every *dia-logos*, the humanistic one appears in trouble, tenuous, if not residual. This sleep of reason is likely to cost us dear:

Oxford is nice, but I wish it weren't dead

wrote the American modern poet, dramatist and essayist Thomas Stearns Eliot, happy for the beautiful university, but worried about the extraneousness and impotence of academics and intellectuals, in the face of the European dramatic situation in pre-world war.
There is a specific responsibility of us: university academics and students. We make the most beautiful job, in the world, because we combine passion and profession; we can say what we think without risking our position. As professors (from the Latin *profiteri*) we are called to profess the truth:

To publicly say all that a search, a knowledge and a thought of truth require (Jacques Derrida [3]).

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[1] Free translation from Italian of the inaugural address at the University of Macerata (Italy) of Ivano Dionigi: The new humanism will save us (La Repubblica, Wednesday April 18th 2018).
[2] Italian famous medieval Italy poet and humanist.
[3] French modern philosopher and essayist.

University students – this is the appeal of the late Umberto Eco [4] – are expected to fight:
the interminable struggle for the progress between knowledge and *pietas*.

From this responsibility can escape: not only the humanists, but also the scientists and technologists, to whom the American entrepreneur, business magnate, inventor and industrial designer Steve Jobs asked for the prerogative of being "Renaissance". Indeed the whole world-society asks them for public responsibility and we (humanists) ask to honor their Ph.D., which means *Philosophiae Doctor*: integral knowledge, increased by long thoughts.
This responsibility calls into question the School and the boys. The School will train digital citizens, as it has in the past trained agricultural citizens, industrial citizens and electronic citizens. There is a paradox:

- while the School is the institution that bears the greatest responsibility, transmitting the fundamentals of knowledge, being the only public social reality where the daily, real, true encounter takes place between peers and between adults and young people;
- unfortunately the School is the most neglected and tormented reality of the country: tormented, because it is affected by permanent reforms, and neglected, because it hasn't adequate social recognition.

The School must be protected and helped. But there is a task that also calls the boys into question. You make the unity, the beauty and the hope of the country blessed rich in talents and damn them regardless of them. This my eyes have seen meeting thousands and thousands of your peers from the deep South to the deep North. Be aware of your grace and your strength, because time is your friend:

every morning when the sun rises, a day begins that has never lived anybody (Davide Maria Turoldo [5]).

Be dissatisfied, be demanding, be strict; I would like to say you are perfect and must dare to know. Engage in politics. do it with passion and pride, not just to affirm yourself; but also do it for a kind of *pietas* towards us, that we didn't make to leave you a better world.

Raffaello Sanzio, La scuola di Atene (Musei Vaticani, Rome)

[4] Italian modern writer and essayist.
[5] Modern Italian poet and theologian, Servite friar.

PART I – SUPPLEMENT DATA PROCESSING

Principal discrete distributions of probability [6]

A. Binomial distribution:

Legend: n size of the sample

k favorable events

Discrete probability density: $p_k = \binom{n}{k} p^k (1-p)^{n-k}$ with $0 < p < 1$

and $k = 0 \div n$

Mean value: $\mu = np$ Mean value of the variable p/n: $\mu = p$

Variance: $\sigma^2 = np(1-p)$ Variance of the variable p/n: $\sigma^2 = \frac{p(1-p)}{n}$

B. Hyper-geometric distribution:

Legend: N size of the universe

r favorable events in the universe

n size of the sample

k favorable events in the sample

Discrete probability density: $p_k = \frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}$ with $k = 0 \div n$

Mean value: $\mu = np$

Variance: $\sigma^2 = np(1-p)\frac{N-n}{N-1}$

C. Geometric distribution:

Legend: k unfortunate events

Discrete probability density: $p_k = (1-p)^k p$ with $0 < p < 1$

[6] At the end of the Elementary transformation paragraph, Cauchy probability distribution is presented, because of the singular behavior in its principal statistics.

and $k = 0 \div ...$

Mean value: $\mu = \dfrac{1-p}{p}$ Variance: $\sigma^2 = \dfrac{1-p}{p^2}$

D. <u>Poisson's distribution:</u>

Legend: k unfortunate events

Discrete probability density: $p_k = e^{-\lambda} \dfrac{\lambda^k}{k!}$ with $\lambda > 0$

and $k = 0 \div ...$

Mean value: $\mu = \lambda$ Variance: $\sigma^2 = \lambda$

E. <u>Negative binomial distribution:</u>

Legend: r favorable events

k unfortunate events

Discrete probability density: $p_k = \begin{pmatrix} r+k-1 \\ k \end{pmatrix} (1-p)^k p^r$ with $0 < p < 1$

and $k = 0 \div ...$

Mean value: $\mu = r \dfrac{1-p}{p}$ Variance: $\sigma^2 = r \dfrac{1-p}{p^2}$

Complementary statistical test of hypothesis

A. <u>Shapiro-Wilk test:</u> Null hypothesis: H_0: $best\ \ fit\ to\ Normal\ distribution$

Alternative hypothesis: H_1

expected value:

$$\hat{w}^2 = \frac{\left(\sum_{i=1}^{m} (x_i - \bar{x}) N^{(-1)} \left(\frac{i-0.5}{m} \right) \right)^2}{\sum_{i=1}^{m} (x_i - \bar{x})^2 \sum_{i=1}^{m} N^{(-1)} \left(\frac{i-0.5}{m} \right)^2}$$

theoretical value: $w^2(\alpha)$

comparison: $\hat{w}^2 \le w^2(\alpha)$ H_0 true

$\hat{w}^2 > w^2(\alpha)$ H_0 false

B. <u>Bresch.Pagan test:</u> Null hypothesis: $H_0: \quad \sigma_i^2 = const \quad \forall i$

Alternative hypothesis: $H_1: \quad \sigma_i^2 \ne const \quad \exists i$

Legend: regression equations: $y_i = a_0 + \sum_{j=1}^{n} a_j x_{ij} + v_i$

equations on the squares of the regression residuals:

$$v_i^2 = w_i = b_0 + \sum_{j=1}^{n} b_j x_{ij} + u_i$$

determination index of the 2nd regression: $R^2 = SSE/SST = 1 - \sum_{i=1}^{N} u_i^2 \Big/ \sum_{i=1}^{N} (w_i - \overline{w})^2$

expected value: $\hat{\chi}_n^2 = NR^2$

theoretical value: $\hat{\chi}_n^2(\alpha)$

comparison: $\hat{\chi}_n^2 \le \hat{\chi}_n^2(\alpha)$ H_0 true

$\hat{\chi}_n^2 > \hat{\chi}_n^2(\alpha)$ H_0 false

C. <u>Scheffé test:</u> Null hypothesis: $H_0: \quad \mu_i = const \quad \forall i$

Alternative hypothesis: $H_1: \quad \mu_i \ne const \quad \exists i$

Legend: weighted mean: $\overline{y} = \sum_{j=1}^{n} a_j \overline{x}_j$

regression mean: $\overline{y} = \hat{a}_0 + \sum_{j=1}^{n} \hat{a}_j \overline{x}_j$

variance of the weighted mean: $\sigma_{\overline{y}}^2 = \hat{\sigma}^2 \sum_{j=1}^{n} a_j^2 / n_j$

variance of the regression mean: $\sigma_{\overline{y}}^2 = \sigma_{\hat{a}_0}^2 + \sum_{j=1}^{n} \overline{x}_j^2 \sigma_{\hat{a}_j}^2 + covariances$

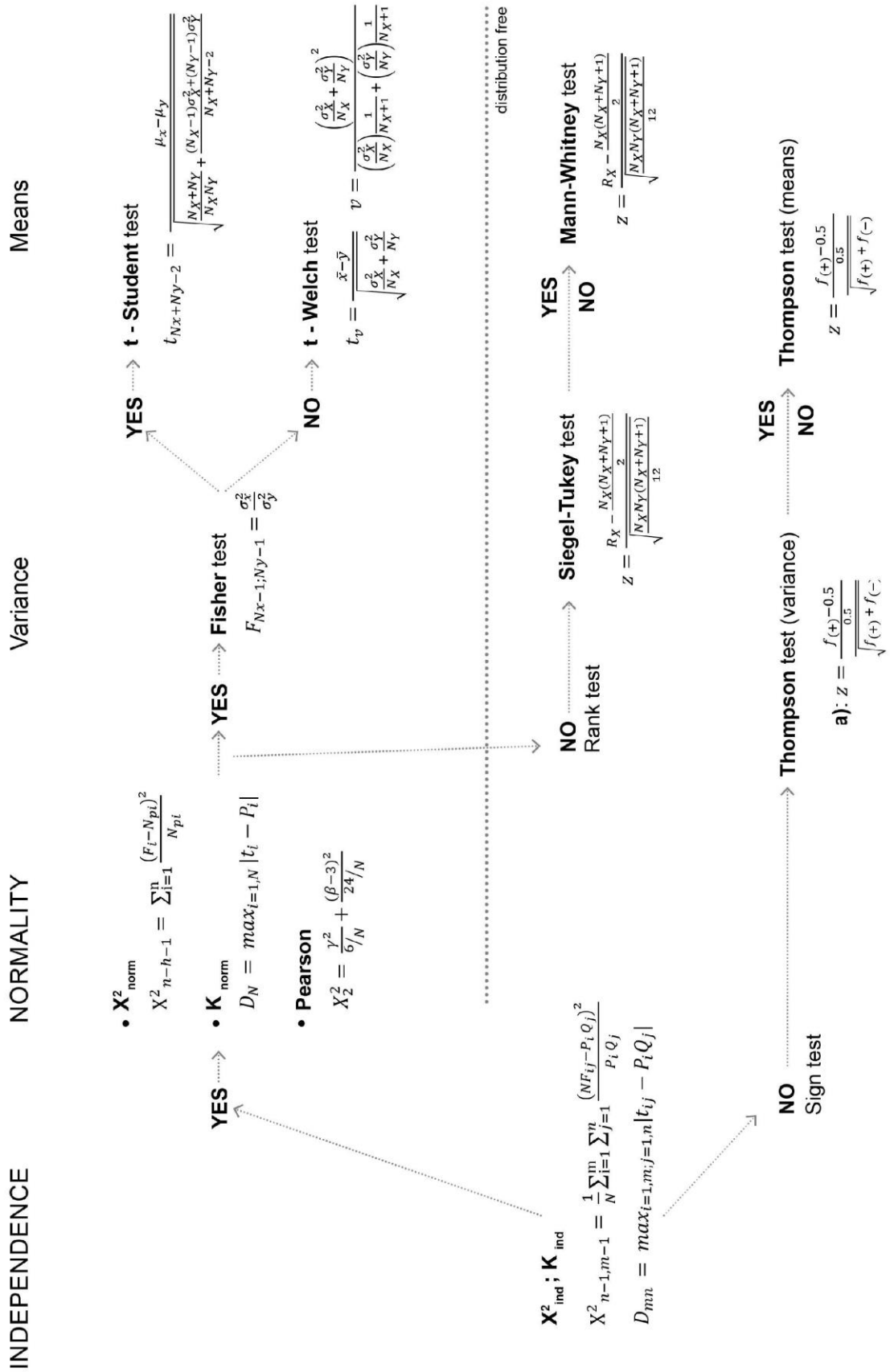
expected value: $\hat{t}_\nu = \overline{y} / \sigma_{\overline{y}}$

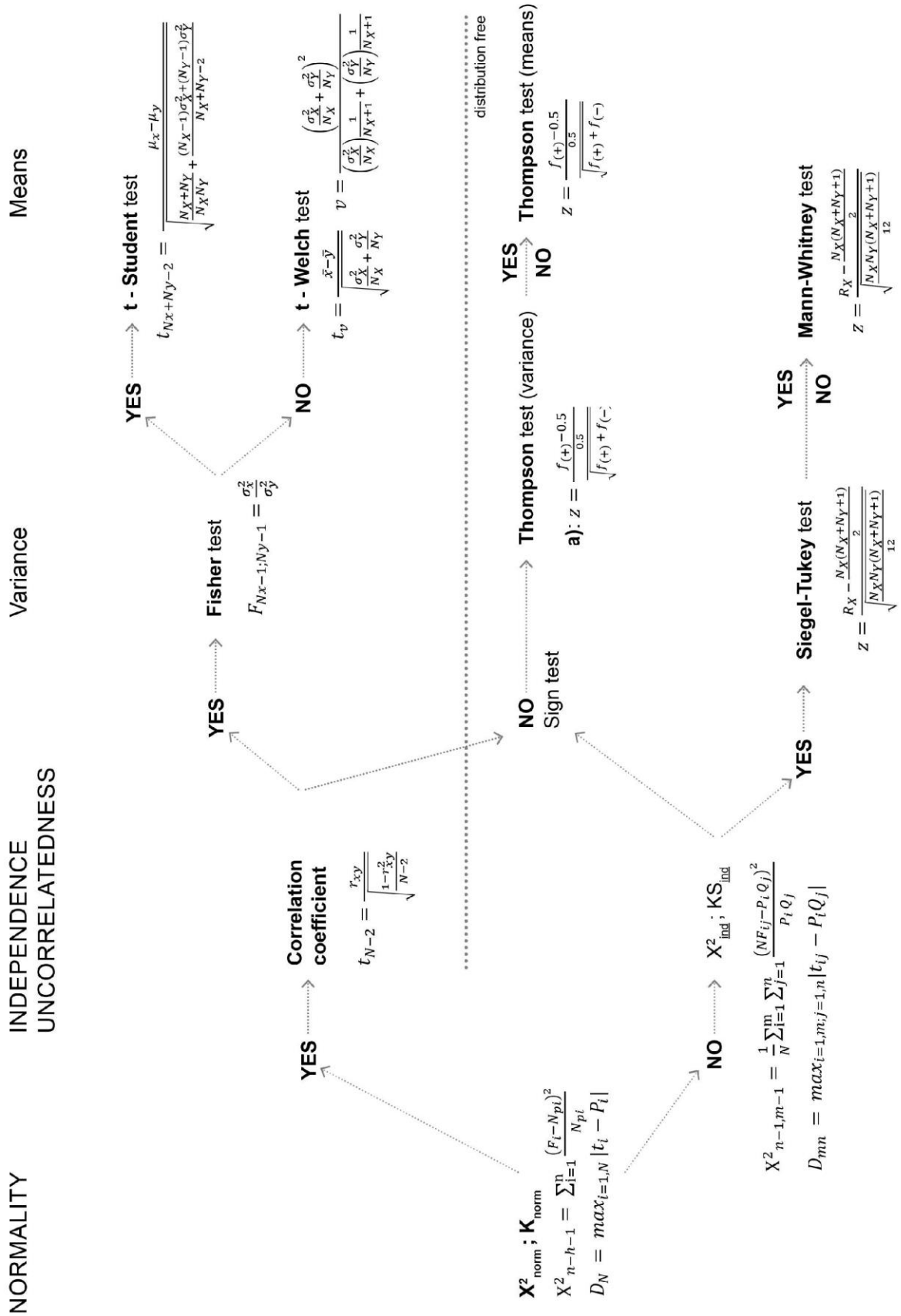
theoretical value: $t_\nu(\alpha)$

comparison: $|\hat{t}_\nu| \le t_\nu(\alpha)$ H_0 true

$|\hat{t}_\nu| > t_\nu(\alpha)$ H_0 false

OPERATIONAL SCHEMES OF STATISTICAL TESTS TO COMPARE TWO SAMPLES

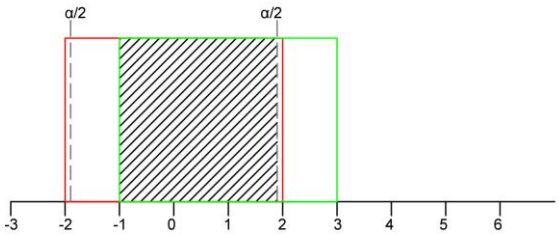




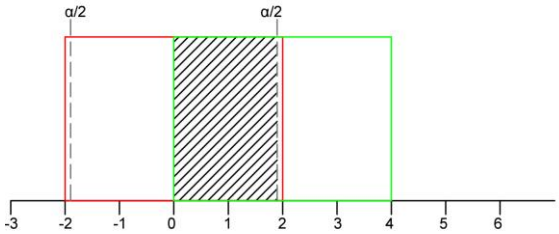
Note: The Mann-Whitney and Siegel-Tukey tests are ranks tests, related respectively to the ordered data and to the deviations from the medians (in absolute value). The Thompson tests on the central values and on the dispersion are tests on the signs, related respectively to the non-nil differences on the data and to the non-nil differences on the deviations from the medians (in absolute value).

APPROXIMATE DETERMINATION OF THE POWER OF A TEST TO BUILD THE POWER CURVE OF STANDARD NORMAL DISTRIBUTION (OR OTHER SYMETRICAL DISTRIBUTIONS FOR WHICH THE HYPOTHESIS ALTERNATIVE IS A TRANSLATION OF THE REFERENCE PARAMETER [7])

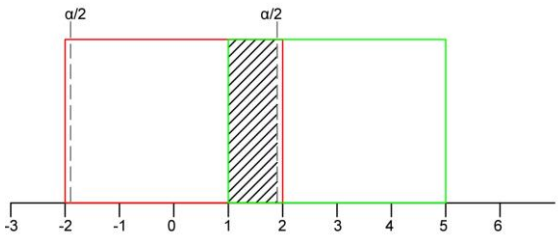
Shift = 1
α = 5% **β ~ 85%**
1-β ~ 15%



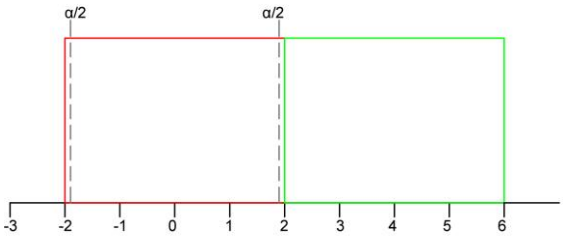
Shift = 2
α = 5% **β ~ 50%**
1-β ~ 50%



Shift = 3
α = 5% **β ~ 15%**
1-β ~ 85%



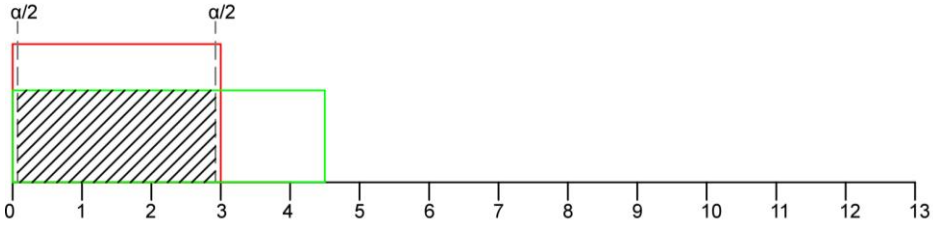
Shift = 4
α = 5% **β ~ 0**
1-β ~ 100%



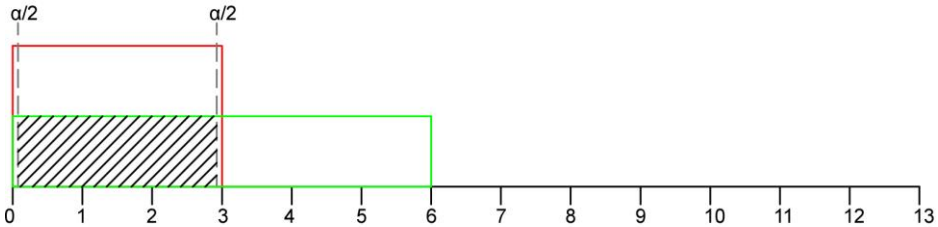
[7] The neutral element of a translation is null, i.e. zero, whilst the neutral element of a scale variation is the unity, i.e. the number one.

APPROXIMATE DETERMINATION OF THE POWER OF A TEST TO BUILD THE POWER CURVE OF THE CHI-SQUARE TYPE DISTRIBUTION [8] (OR OTHER "POSITIVE" DISTRIBUTES FOR WHICH THE HYPOTHESIS ALTERNATIVE IS A SCALE VARIATION OF THE REFERENCE PARAMETER)

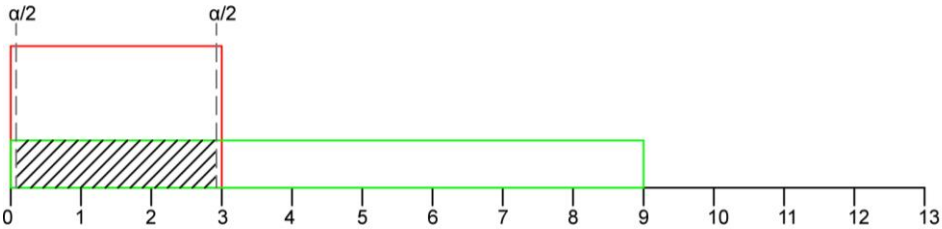
Scale variation = 1.5
α = 5% **β ~ 66.6%**
1-β ~ 33.3%



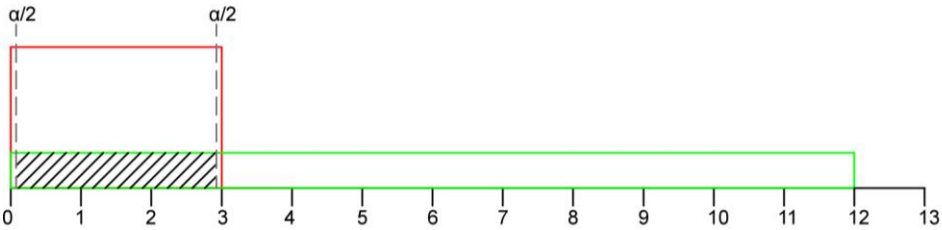
Scale variation = 2
α = 5% **β ~ 50%**
1-β ~ 50%



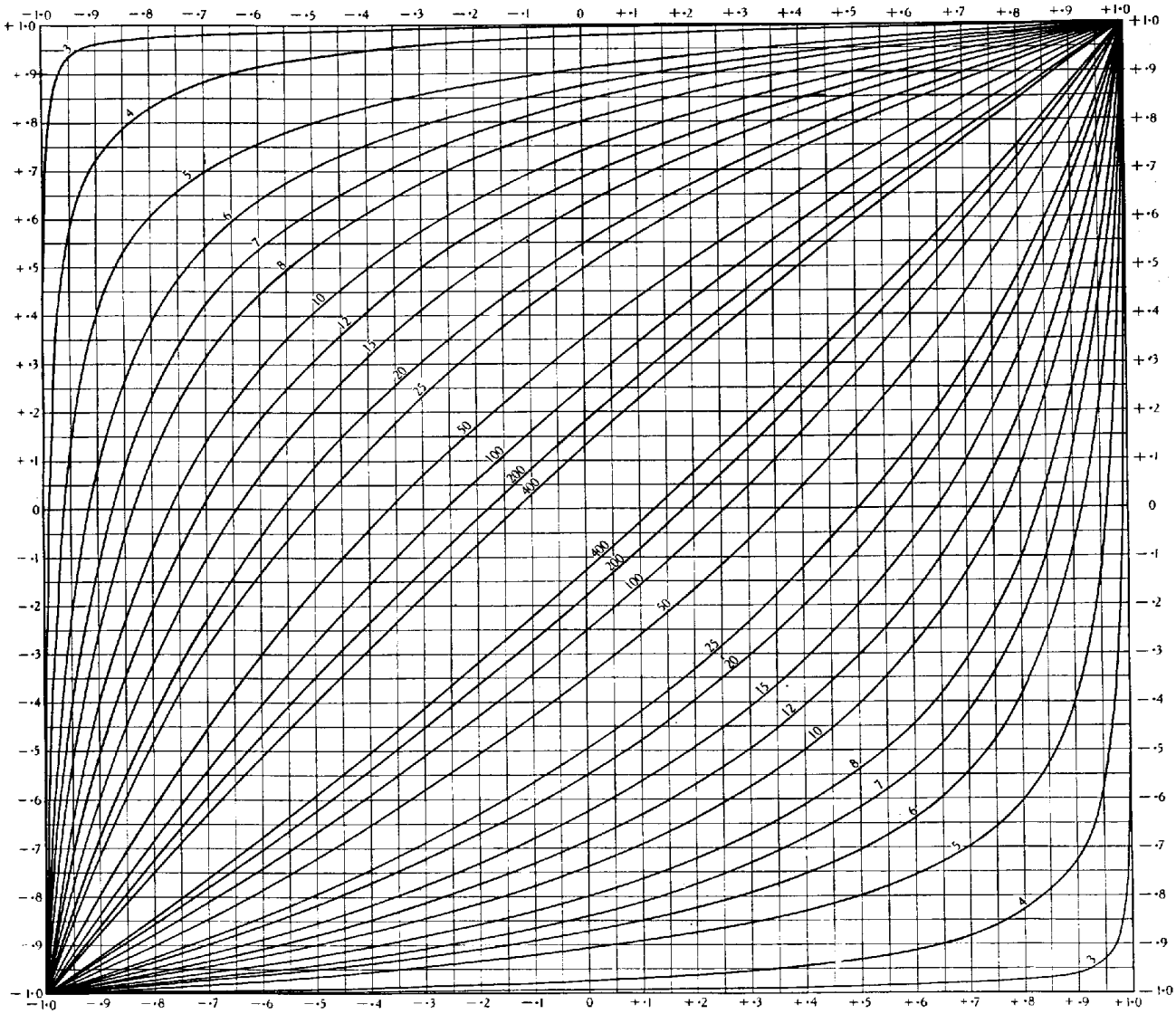
Scale variation = 3
α = 5% **β ~ 33.3%**
1-β ~ 66.6%



Scale variation = 4
α = 5% **β ~ 25%**
1-β ~ 75%



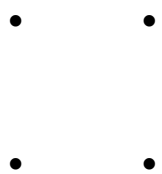
[8] The chi-square distribution depends on its degrees of freedom; therefore this example is purely qualitative.



Confidence bounds of the correlation coefficient

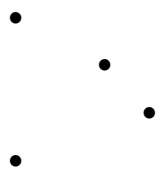
Data modeling about dependence conditions and error types

1. <u>Independence:</u>



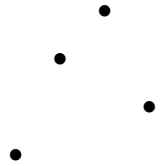
where the Bonferroni's, Pearson's indices and correlation coefficient are equal to zero.

2. <u>Connection:</u>



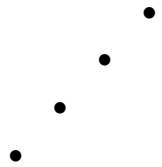
where Bofferroni's indices are different from zero, whilst Pearson's indices and correlation coefficient are near to zero.

3. Regression:



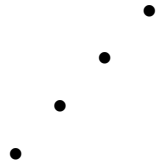
where Bofferroni's and Pearson's indices are different from zero, whilst correlation coefficient are near to zero.

4. Correlation:



where all the indices are different from zero.

5. Perfect dependence in law:

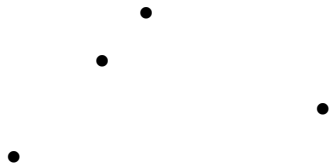


where all the indices are equal to one.

A. An outlier [9]:

which asks for down-weighting robust procedures.

B. A leverage point:



which implies robust procedures with a very high break-down and a rejection point in their influence functions.

[9] Observations are usually supposed error free; otherwise two type of error can occur in the data.

Elementary transformations

2D case

1. S transformation:

$$X = \Delta X + x\lambda \cos\vartheta + y\lambda \sin\vartheta \qquad X = \Delta X + xa + yb$$
$$Y = \Delta Y - x\lambda \sin\vartheta + y\lambda \cos\vartheta \qquad Y = \Delta Y - xb + ya$$

being $a = \lambda \cos\vartheta$ and $b = \lambda \sin\vartheta$

so that:

$$a = \frac{E(xX) + E(yY)}{E(x^2) + E(y^2)} \qquad \text{and} \qquad \vartheta = arc\tan\frac{b}{a}$$
$$b = \frac{E(yX) - E(xY)}{E(x^2) + E(y^2)} \qquad \lambda = \sqrt{a^2 + b^2}$$

(note: here and in the follows: $x = x - \bar{x}$ $\quad y = y - \bar{y}$
$X = X - \bar{X}$ $\quad Y = Y - \bar{Y}$)

$$\Delta X = \bar{X} - \bar{x}a - \bar{y}b$$
$$\Delta Y = \bar{Y} + \bar{x}b - \bar{y}a$$

2. Rotary-translation:

$$X = \Delta X + x \cos\vartheta + y \sin\vartheta$$
$$Y = \Delta Y - x \sin\vartheta + y \cos\vartheta$$

is solved using the same formulas, imposing a scale equal to one, because the S-transformation preserves the angles. (being: $\sigma_{ab} = 0$, and therefore: $\sigma_{\lambda\vartheta} = 0$).

3. Affine transformation:

$$X = \Delta X + xa + yb \qquad \text{or:} \qquad X = \Delta X + x\lambda \cos\vartheta + y\lambda \sin\vartheta$$
$$Y = \Delta Y + xc + yd \qquad\qquad Y = \Delta Y - x\mu \sin(\vartheta + \gamma) + y\mu \cos(\vartheta + \gamma)$$

analogously:

$$a = \frac{E(y^2)E(xX) - E(xy)E(yX)}{E(x^2)E(y^2) - (E(xy))^2}$$
$$b = \frac{-E(xy)E(xX) + E(x^2)E(yX)}{E(x^2)E(y^2) - (E(xy))^2}$$
$$c = \frac{E(y^2)E(xY) - E(xy)E(yY)}{E(x^2)E(y^2) - (E(xy))^2}$$
$$d = \frac{-E(xy)E(xY) + E(x^2)E(yY)}{E(x^2)E(y^2) - (E(xy))^2}$$

being: $\Delta X = \bar{X} - \bar{x}a - \bar{y}b \qquad \Delta Y = \bar{Y} - \bar{x}c - \bar{y}d$

$a = \lambda \cos\vartheta \qquad b = \lambda \sin\vartheta \qquad c = -\mu \sin(\vartheta + \gamma) \qquad d = \mu \cos(\vartheta + \gamma)$

$$\vartheta = arctan\, b/a \qquad \gamma = arctan(-c/d) - \vartheta$$
$$\lambda = \sqrt{a^2 + b^2} \qquad \mu = \sqrt{c^2 + d^2}$$

4. <u>Particular affine transformation:</u>

$$X = \Delta X + x\lambda \cos\vartheta + y\lambda \sin\vartheta \qquad X = xa + yb$$
$$Y = \Delta Y - x\mu \sin\vartheta + y\mu \cos\vartheta \qquad Y = xc + yd$$

additional constrain: $ac + bd = 0$

being: $a = \lambda \cos\vartheta \quad b = \lambda \sin\vartheta \quad c = \mu \cos\vartheta \quad d = \mu \sin\vartheta$

i.e.:

$$s - Rz = 0 \qquad \text{and} \qquad z^T \Gamma\, z = 0$$

where:

$$R = \begin{bmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

least squares method and iterative solution:

$$g - (C + k\Gamma)z = 0$$
$$z^T \Gamma\, z = 0$$

being: $g = \sum_i R_i^T s_i$ and $C = \sum_i R_i^T R_i$

$$z = C^{-1}(g - k\Gamma z)$$
$$\left(z^T \Gamma C^{-1} \Gamma C^{-1} \Gamma z\right) k^2 + 2\left(g C^{-1} \Gamma C^{-1} \Gamma z\right) k + g C^{-1} \Gamma C^{-1} g = 0$$

(note: $\tilde{k} = 0$
$k = min(k_1, k_2)$)

so that:

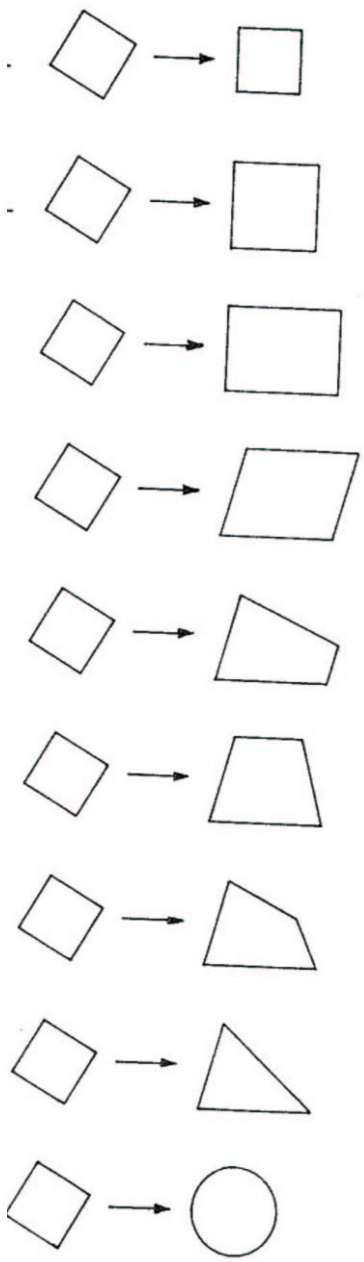
$$a = \frac{E(y^2)(E(xX) - kc) - E(xy)(E(yX) - kd)}{E(x^2)E(y^2) - (E(xy))^2}$$

$$b = \frac{-E(xy)(E(xX) - kc) + E(x^2)(E(yX) - kd)}{E(x^2)E(y^2) - (E(xy))^2}$$

$$c = \frac{E(y^2)(E(xY) - ka) - E(xy)(E(yY) - kb)}{E(x^2)E(y^2) - (E(xy))^2}$$

$$d = \frac{-E(xy)(E(xY) - ka) + E(x^2)(E(yY) - kb)}{E(x^2)E(y^2) - (E(xy))^2}$$

furthermore, being: $k = \dfrac{V - \sqrt{V^2 - UW}}{U}$



Rotary-translation

S transformation

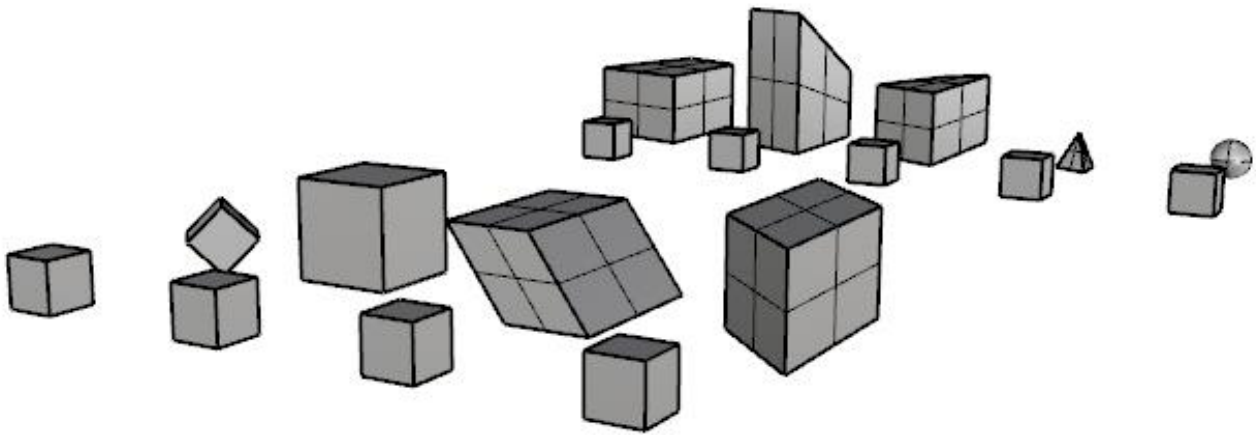
Particular affine transformation

Affine transformation

Homographic transformations

Topological transformations

Plane elementary transformations



Spatial elementary transformation examples

$$U = 2\left(E(xy)^2 + E\left(y^2\right)^2\right)ac - 2\left(E\left(x^2\right) + E\left(y^2\right)\right)E(xy)(ad + bc) + 2\left(E\left(x^2\right)^2 + E(xy)^2\right)bd$$

$$V = \left(E(xy)^2 + E\left(y^2\right)^2\right)(aE(xX) + cE(xY)) - \left(E\left(x^2\right) + E\left(y^2\right)\right)E(xy)\cdot$$
$$\cdot(bE(xX) + dE(xY) + aE(yX) + cE(yY)) + \left(E\left(x^2\right)^2 + E(xy)^2\right)(bE(yX) + dE(yY))$$

$$W = 2\left(E(xy)^2 + E\left(y^2\right)^2\right)E(xX)E(xY) - 2\left(E\left(x^2\right) + E\left(y^2\right)\right)E(xy)(E(xX)E(yY) + E(xY)E(yX)) +$$
$$+ 2\left(E\left(x^2\right)^2 + E(xy)^2\right)E(yX)E(yY)$$

5. <u>Homographic transformation:</u>

$$X = \frac{ax + by + c}{gx + hy + 1}$$

$$Y = \frac{dx + ey + f}{gx + hy + 1}$$

forming the normal matrix and the normal know-vector:

$$C = \begin{bmatrix} E\left(x^2\right) & E(xy) & E(x) & 0 & 0 & 0 & -E\left(x^2X\right) & -E(xyX) \\ & E\left(y^2\right) & E(y) & 0 & 0 & 0 & -E(xyX) & -E\left(y^2X\right) \\ & & 1 & 0 & 0 & 0 & -E(xX) & -E(yX) \\ & & & E\left(x^2\right) & E(xy) & E(x) & -E\left(x^2Y\right) & -E(xyY) \\ & & & & E\left(y^2\right) & E(y) & -E(xyY) & -E\left(y^2Y\right) \\ & & & & & 1 & -E(xY) & -E(yY) \\ & sim. & & & & & E\left(x^2X^2\right) + E\left(y^2X^2\right) & E\left(xyX^2\right) + E\left(xyY^2\right) \\ & & & & & & & E\left(y^2X^2\right) + E\left(y^2Y^2\right) \end{bmatrix}$$

$$d^T = \left[E(xX) \quad E(yX) \quad E(X) \quad E(xY) \quad E(yY) \quad E(Y) \quad -E\left(xX^2\right) - E\left(xY^2\right) \quad -E\left(yX^2\right) - E\left(yY^2\right)\right]$$

being: $X = ax + by + c - gxX - hyX$ $\qquad$ $Y = ax + by + c - gxY - hyY$

the normal system is easily solved; then:

$$\Delta X = c \qquad \Delta Y = f$$
$$\alpha = arctan(g\cos\beta) \cong g \qquad \beta = arctan(-h) \cong -h$$

whilst the remaining parameters are obtained from the same formulas of the affine transformation.

<u>3D case</u>

1. <u>S transformation:</u>

$$y = \lambda Rx + t$$
$$\underline{\bar{y} = \lambda R\bar{x} + t}$$
$$y - \bar{y} = \lambda R(x - \bar{x})$$

so that: $t = \bar{y} - \lambda R\bar{x}$ $\qquad$ and $\qquad$ $y^T y = \lambda^2 x^T R^T Rx = \lambda^2 x^T x \Rightarrow \lambda = \sqrt{y^T y / x^T x}$

therefore:

$$y = \lambda R x \qquad \text{(note: } x = \lambda x\text{)}$$

$$y = Rx = (I - S)^{-1}(I + S)x \;\Rightarrow\; (I - S)y = (I + S)x$$

being:

$$R = (I - S)^{-1}(I + S) \qquad \text{and} \qquad S = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

so that:

$$\begin{bmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i.e.:

$$\begin{bmatrix} 0 & Z - z & -(Y - y) \\ -(Z - z) & 0 & X - x \\ Y - y & -(X - x) & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} X - x \\ Y - y \\ Z - z \end{bmatrix} = 0$$

Rodriguez's matrix:

$$R = \frac{1}{1 + a^2 + b^2 + c^2}\begin{bmatrix} 1 + a^2 - b^2 - c^2 & 2(ab - c) & 2(ac + b) \\ 2(ab + c) & 1 - a^2 + b^2 - c^2 & 2(bc - a) \\ 2(ac - b) & 2(bc + a) & 1 - a^2 - b^2 + c^2 \end{bmatrix}$$

Cardanus' matrix:

$$R = \begin{bmatrix} \cos k & \sin k & 0 \\ -\sin k & \cos k & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\varphi\cos k & \cos\omega\sin k + \sin\omega\sin\varphi\cos k & \sin\omega\sin k - \cos\omega\sin\varphi\cos k \\ -\cos\varphi\sin k & \cos\omega\cos k - \sin\omega\sin\varphi\sin k & \sin\omega\cos k + \cos\omega\sin\varphi\sin k \\ \sin\varphi & -\sin\omega\cos\varphi & \cos\omega\cos\varphi \end{bmatrix}$$

Euler's matrix:

$$R = \begin{bmatrix} \cos k & \sin k & 0 \\ -\sin k & \cos k & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{bmatrix}\begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\vartheta\cos\alpha - \sin\vartheta\cos\zeta\sin\alpha & \sin\vartheta\cos\alpha + \cos\vartheta\cos\zeta\sin\alpha & \sin\alpha\sin\zeta \\ -\cos\vartheta\sin\alpha - \sin\vartheta\cos\zeta\cos\alpha & -\sin\vartheta\sin\alpha + \cos\vartheta\cos\zeta\cos\alpha & \cos\alpha\sin\zeta \\ \sin\vartheta\sin\zeta & -\cos\vartheta\sin\zeta & \cos\zeta \end{bmatrix}$$

Cardanus' angles:

- ❑ φ from r_{31} by means of $arc\ sin\ r_{31}$
- ❑ k r_{11} and r_{21} $arc\,tan(-r_{21}/r_{11})$
- ❑ ω r_{32} r_{33} $arc\,tan(-r_{32}/r_{33})$

(note: if $\varphi = \pm\pi/2$ $\kappa \equiv 0$
then ω from r_{22} and r_{23} by means of $arctan(r_{23}/r_{22})$)

Euler's angles:

- ❑ ζ from r_{33} by means of $arc\ cos\ r_{33}$
- ❑ ϑ r_{31} and $-r_{32}$ $arctan(r_{31}/-r_{32})$
- ❑ α r_{13} r_{23} $arctan(r_{13}/r_{23})$

(note: if $\zeta = 0$ or $\zeta = \pi$ $\alpha \equiv 0$
then ϑ from r_{12} and r_{11} by means of $arctan(r_{12}/r_{11})$)

2. <u>Rotary-translation:</u>
(Note: the parameters are obtained from the same formulas of the S transformation, imposing: $\lambda \equiv 1$)

3. Affine transformation:

$$X = \Delta X + xa + yb + zc$$
$$Y = \Delta Y + xd + ye + zf$$
$$Z = \Delta Z + xg + yh + zl$$

being:

$$a = \frac{AE(xX) - BE(yX) + CE(zX)}{H} \qquad b = \frac{-BE(xX) + DE(yX) - FE(zX)}{H}$$
$$c = \frac{CE(xX) - FE(yX) + GE(zX)}{H}$$
$$d = \frac{AE(xY) - BE(yY) + CE(zY)}{H} \qquad e = \frac{-BE(xY) + DE(yY) - FE(zY)}{H}$$
$$f = \frac{CE(xY) - FE(yY) + GE(zY)}{H}$$
$$g = \frac{AE(xZ) - BE(yZ) + CE(zZ)}{H} \qquad h = \frac{-BE(xZ) + DE(yZ) - FE(zZ)}{H}$$
$$k = \frac{CE(xZ) - FE(yZ) + GE(zZ)}{H}$$

where:

$$A = E(y^2)E(z^2) - (E(yz))^2 \qquad B = E(xy)E(z^2) - E(zx)E(yz)$$
$$D = E(x^2)E(z^2) - (E(zx))^2 \qquad C = E(xy)E(yz) - E(zx)E(y^2)$$
$$G = E(x^2)E(y^2) - (E(xy))^2 \qquad F = E(x^2)E(yz) - E(zx)E(xy)$$

$$H = E(x^2)E(y^2)E(z^2) + 2E(xy)E(yz)E(zx) - E(x^2)(E(yz))^2 - E(y^2)(E(zx))^2 - E(z^2)(E(xy))^2$$

so that:

$$\Delta X = \bar{X} - \bar{x}a - \bar{y}b - \bar{z}c \qquad \lambda = \sqrt{a^2 + b^2 + c^2}$$
$$\Delta Y = \bar{Y} - \bar{x}d - \bar{y}e - \bar{z}f \qquad \mu = \sqrt{d^2 + e^2 + f^2}$$
$$\Delta Z = \bar{Z} - \bar{x}g - \bar{y}h - \bar{z}l \qquad \nu = \sqrt{g^2 + h^2 + l^2}$$

moreover:

$$Q = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & L \end{bmatrix}$$

where:

$$A = cos(\varphi + \delta) cos\, k \qquad B = cos\omega\, sin\, k + sin\, \omega\, sin(\varphi + \delta) cos\, k$$
$$C = sin\, \omega\, sin\, k - cos\, \omega\, sin(\varphi + \delta) cos\, k$$
$$D = -cos\, \varphi\, sin(k + \gamma) \qquad E = cos\, \omega\, cos(k + \gamma) - sin\, \omega\, sin\, \varphi\, sin(k + \gamma)$$
$$F = sin\, \omega\, cos(k + \gamma) + cos\, \omega\, sin\, \varphi\, sin(k + \gamma)$$
$$G = sin\, \varphi \qquad K = -sin(\omega + \varepsilon) cos\, \varphi$$
$$L = cos(\omega + \varepsilon) cos\, \varphi$$

so that:

$$\varphi = arc\, sin\, G \qquad \omega + \varepsilon = arc\, tan \frac{-H}{L}$$
$$k + \gamma = arcsin \frac{-D}{cos\, \varphi}$$

$$\omega = arc\, sin \frac{Fcos(k + \gamma) - Esin\varphi\, sin(k + \gamma)}{cos^2(k + \gamma) + sin^2\varphi\, sin^2(k + \gamma)} = arc\, cos \frac{Ecos(k + \gamma) + Fsin\varphi\, sin(k + \gamma)}{cos^2(k + \gamma) + sin^2\varphi\, sin^2(k + \gamma)}$$

$$k = arc\, sin(Bcos\omega + Csin\omega) \qquad \varphi + \delta = arccos \frac{A}{cos\, k}$$

4. <u>Particular affine transformation:</u>

$$X = \Delta X + x\lambda\, cos\, \varphi\, cos\, k + y\lambda\, (cos\, \omega\, sink + sin\, \omega\, sin\, \varphi\, cos\, k) + z\lambda\, (sin\, \omega\, sink - cos\, \omega\, sin\, \varphi\, cos\, k)$$
$$Y = \Delta Y - x\mu\, cos\, \varphi\, sink + y\mu\, (cos\, \omega\, cos\, k - sin\, \omega\, sin\, \varphi\, sink) + z\mu\, (sin\, \omega\, cos\, k + cos\, \omega\, sin\, \varphi\, sink)$$
$$Z = \Delta Z + x\nu\, sin\, \varphi - y\nu\, sin\, \omega\, cos\, \varphi + z\nu\, cos\, \omega\, cos\, \varphi$$

$$X = xa + yb + zc$$
$$Y = xd + ye + zf$$
$$Z = xg + yh + zl$$

additional constrains:

$$ad + be + cf = 0$$
$$ag + bh + cl = 0$$
$$dg + eh + fl = 0$$

i.e.:

$$s - Rz = 0$$

$$z^T \Gamma_1 z = 0 \qquad z^T \Gamma_2 z = 0 \qquad z^T \Gamma_3 z = 0$$

where:

$$R = \begin{bmatrix} x & y & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & y & z \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \qquad \Gamma_3 = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$$

least squares method and iterative solution (taking into account a 6th degree equation):

$$g - \left(C + k_1\Gamma_1 + k_2\Gamma_2 + k_3\Gamma_3\right)z = 0 \qquad z = C^{-1}\left(g - \left(k_1\Gamma_1 + k_2\Gamma_2 + k_3\Gamma_3\right)z\right)$$
$$z^T \Gamma_1 z = 0 \qquad z^T \Gamma_2 z = 0 \qquad z^T \Gamma_3 z = 0$$

being: $g = \sum_i R_i^T s_i$ and $C = \sum_i R_i^T R_i$

5. <u>Homographic transformation:</u>

$$X = \frac{ax + by + cz + d}{px + qy + rz + 1}$$
$$Y = \frac{ex + fy + gz + h}{px + qy + rz + 1}$$
$$Z = \frac{kx + ly + mz + n}{px + qy + rz + 1}$$

i.e.:

$$X = ax + by + cz + d - pxX - qyX - rzX$$
$$Y = ex + fy + gz + h - pxY - qyY - rzY$$
$$Z = kx + ly + mz + n - pxZ - qyZ - rzZ$$

forming the normal matrix and the normal know-vector:

$$C = \begin{bmatrix} T & 0 & 0 & U \\ & T & 0 & V \\ & & T & W \\ sim. & & & S \end{bmatrix}$$

(note: the last matrix is the matrix S)

$$T = \begin{bmatrix} E(x^2) & E(xy) & E(xz) & E(x) \\ & E(y^2) & E(yz) & E(y) \\ & & E(z^2) & E(z) \\ & & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -E(x^2X) & -E(xyX) & -E(xzX) \\ -E(xyX) & -E(y^2X) & -E(yzX) \\ -E(xzX) & -E(yzX) & -E(z^2X) \\ -E(xX) & E(yX) & E(zX) \end{bmatrix}$$

$$V = \begin{bmatrix} -E(x^2Y) & -E(xyY) & -E(xzY) \\ -E(xyY) & -E(y^2Y) & -E(yzY) \\ -E(xzY) & -E(z^2Y) & -E(z^2Y) \\ -E(xY) & E(yY) & E(zY) \end{bmatrix} \qquad W = \begin{bmatrix} -E(x^2Z) & -E(xyZ) & -E(xzZ) \\ -E(xyZ) & -E(y^2Z) & -E(yzZ) \\ -E(xzZ) & -E(yzZ) & -E(z^2Z) \\ -E(xZ) & E(yZ) & E(zZ) \end{bmatrix}$$

$$\begin{bmatrix} E(x^2X^2)+E(y^2X^2)+E(z^2X^2) & E(xyX^2)+E(xyY^2)+E(xyZ^2) & E(xzX^2)+E(xzY^2)+E(xzZ^2) \\ & E(x^2Y^2)+E(y^2Y^2)+E(z^2Y^2) & E(yzX^2)+E(yzY^2)+E(yzZ^2) \\ & & E(x^2Z^2)+E(y^2Z^2)+E(z^2Z^2) \end{bmatrix}$$

$d^T = [u^T \quad v^T \quad w^T \quad s^T]$ (note: the last vector is the vector s^T)

$u^T = [E(xX) \quad E(yX) \quad E(zX) \quad E(X)]$ $\qquad v^T = [E(xY) \quad E(yY) \quad E(zY) \quad E(Y)]$

$w^T = [E(xZ) \quad E(yZ) \quad E(zZ) \quad E(Z)]$

$$[-E(xX^2)-E(yX^2)-E(zX^2) \quad -E(xY^2)-E(yY^2)-E(zY^2) \quad -E(xZ^2)-E(yZ^2)-E(zZ^2)]$$

the normal system is easily solved; then:

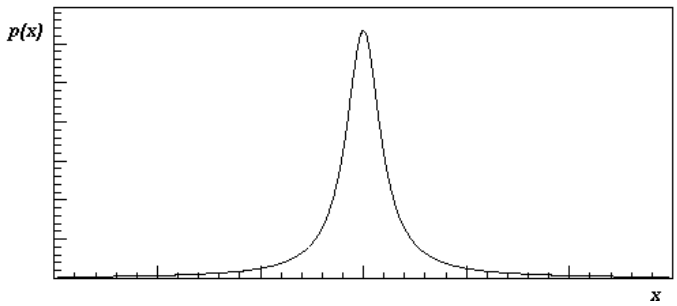
$\Delta X = d \qquad \Delta Y = h \qquad \Delta Z = n$

angles and scales are obtained as usual, and the solid angles suitably applying the following formula:

$$\Omega = 2\,arc\,tan \frac{det(R_1R_2R_3)}{R_1R_2R_3 + (R_1 \cdot R_2)R_3 + (R_2 \cdot R_3)R_1 + (R_3 \cdot R_1)R_2}$$

Cauchy probability distribution

$p(x) = \dfrac{1}{\pi(1-x^2)}$ where: $p(0) = 1/\pi \cong 0.32$



Generalized mean value and variance:

$\gamma(x)$ so that $E_x(arctan(x-\gamma)) = 0$

$\delta(x) = -log\, E_x\left(E_y\left(e^{-|x-y|}\right)\right)$ $\qquad \forall x, y \neq x$

Random variable transformation: **mean value theorem variance covariance propagation law**

- One-dimensional case: $y = g(x)$:

<u>input:</u>

$$\bar{x} = E(x\ p(x)) \qquad \bar{y} = E(y\ p(y))$$
$$\sigma_x^2 = E\left((x-\bar{x})^2\ p(x)\right) \qquad \sigma_y^2 = E\left((y-\bar{y})^2\ p(y)\right)$$
$$y = g(x) \cong g(\bar{x}) + g'(\bar{x})(x-\bar{x})$$

(note: $E(p(x)) = 1 \quad E((x-\bar{x})p(x)) = 0$)

<u>output:</u>

$$\bar{y} = E(y\ p(y)) \cong g(\bar{x})E(p(x)) + g'(\bar{x})E((x-\bar{x})p(x)) = g(\bar{x})$$
$$\sigma_y^2 = E\left((y-\bar{y})^2\ p(y)\right) \cong E\left((g(\bar{x}) + g'(\bar{x})(x-\bar{x}) - g(\bar{x}))^2\ p(x)\right) = g'(\bar{x})^2 E\left((x-\bar{x})^2\ p(x)\right) =$$
$$= g'(\bar{x})^2 \sigma_x^2$$

- Multi-dimensional case: $Y = G(X)$:

<u>input:</u>

$$\bar{X} = E(X\ p(X)) \qquad \bar{Y} = E(Y\ p(Y))$$
$$C_{XX} = E\left((X-\bar{X})(X-\bar{X})^T\ p(X)\right) \qquad C_{YY} = E\left((Y-\bar{Y})(Y-\bar{Y})^T\ p(Y)\right)$$
$$Y = G(X) \cong G(\bar{X}) + J(\bar{X})(X-\bar{X})$$

(note: $J = \lfloor \partial g_i / \partial x_j \rfloor \ \forall i, j$ Jacobian matrix)

<u>output:</u>

$$\bar{Y} = E(Y\ p(Y)) \cong G(\bar{X})E(p(X)) + J(\bar{X})E((X-\bar{X})p(X)) = G(\bar{X})$$
$$C_{YY} = E\left((Y-\bar{Y})(Y-\bar{Y})^T\ p(Y)\right) \cong E\left(\left(G(\bar{X}) + J(\bar{X})(X-\bar{X})(X-\bar{X})^T J(\bar{X})^T - G(\bar{X})\right)^2 p(x)\right)$$
$$= J(\bar{X})E\left((X-\bar{X})(X-\bar{X})^T\ p(x)\right)J(\bar{X})^T = J(\bar{X})C_{XX}J(\bar{X})^T$$

and in the linear case: ($Y = AX + B$):

$$\bar{Y} = A\bar{X} + B \qquad C_{YY} = AC_{XX}A^T$$

i.e. algebraically:

$$\bar{y} = \sum_{i=1}^{n} a_i \bar{x}_i + b \qquad \forall y$$
$$\sigma_y^2 = \sum_{i=1}^{n} a_i^2 \sigma_{x_i}^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \sigma_{x_i x_j} \qquad \forall y$$
$$\sigma_{yz}^2 = \sum_{i=1}^{n} a_i b_i \sigma_{x_i}^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j + b_i a_j)\sigma_{x_i x_j} \qquad \forall y, z > y$$

being:

$$C_{YY} = AC_{XX}A^T \quad \Rightarrow \quad vec(C_{YY}) = (A \otimes A)vec(C_{XX})$$

(note: $A \otimes B = \lfloor a_{ij} B \rfloor \quad \forall i, j$)

- Measurements of an example:
  - base line: a
  - height: b
- Observation variances: σ_a^2 ; σ_b^2 (note: the measurements are usually independent)
- Estimation quantities: (partial derivates)
  - perimeter: $2p = 2a + 2b$ $\quad \frac{\partial(2p)}{\partial a} = 2$ $\quad \frac{\partial(2p)}{\partial b} = 2$
  - Area: $A = ab$ $\quad \frac{\partial A}{\partial a} = b$ $\quad \frac{\partial A}{\partial b} = a$
  - diagonal: $d = \sqrt{a^2 + b^2}$ $\quad \frac{\partial d}{\partial a} = \frac{a}{d}$ $\quad \frac{\partial d}{\partial b} = \frac{b}{d}$
  - tangent of the angle: $t = \tan \vartheta = b/a$ $\quad \frac{\partial t}{\partial a} = -\frac{t}{a}$ $\quad \frac{\partial t}{\partial b} = \frac{t}{b}$
- Estimation variances:
  - perimeter: $\sigma_{2p}^2 = 2^2 \cdot \sigma_a^2 + 2^2 \cdot \sigma_b^2$
  - Area: $\sigma_A^2 = b^2 \cdot \sigma_a^2 + a^2 \cdot \sigma_b^2$
  - diagonal: $\sigma_d^2 = \left(\frac{a}{d}\right)^2 \sigma_a^2 + \left(\frac{b}{d}\right)^2 \sigma_b^2$
  - tangent of the angle: $\sigma_t^2 = \left(-\frac{t}{a}\right)^2 \sigma_a^2 + \left(\frac{t}{b}\right)^2 \sigma_b^2$
- Estimation covariances:
  - perimeter/Area: $\sigma_{2pA} = 2b\sigma_a^2 + 2a\sigma_b^2$
  - perimeter/diagonal: $\sigma_{(2p)d} = 2\frac{a}{d}\sigma_a^2 + 2\frac{b}{d}\sigma_b^2$
  - perimeter/tangent: $\sigma_{(2p)t} = 2\frac{t}{b}\sigma_a^2 + 2\left(-\frac{t}{a}\right)\sigma_b^2$
  - Area/diagonal: $\sigma_{Ad} = b\frac{a}{d}\sigma_a^2 + a\frac{b}{d}\sigma_b^2$
  - Area/tangent: $\sigma_{At} = b\left(-\frac{t}{a}\right)\sigma_a^2 + a\frac{t}{b}\sigma_b^2$
  - diagonal/tangent $\sigma_{dt} = \frac{a}{d}\left(-\frac{t}{a}\right)\sigma_a^2 + \frac{b}{d}\frac{t}{b}\sigma_b^2$

Complementary descriptive statistics

- One-dimensional case:

mean value: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ variance: $\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

variance of the mean value: $\sigma_{\bar{x}}^2 = \dfrac{\sigma^2}{n}$ var. of the variance: $\sigma_{\sigma^2}^2 = \dfrac{2\sigma^4}{n}$ [10]

condition number: $\chi_\infty = n^2$ local redundancy: $\nu = 1 - 1/n$

- Two-dimensional case:

regression lines: $a = \dfrac{\sigma_{xy}}{\sigma_x^2}$ and $c = \dfrac{\sigma_{xy}}{\sigma_y^2}$

$b = \bar{y} - a\bar{x}$ $d = \bar{x} - c\bar{y}$

covariance matrices:

$$\begin{bmatrix} \sigma_a^2 & \sigma_{ab} \\ \bullet & \sigma_b^2 \end{bmatrix} = \sigma_0^2 \begin{bmatrix} \dfrac{1}{n\sigma_x^2} & -\dfrac{\bar{x}}{n\sigma_x^2} \\ \bullet & \dfrac{1}{n} + \dfrac{\bar{x}^2}{n\sigma_x^2} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_c^2 & \sigma_{cd} \\ \bullet & \sigma_d^2 \end{bmatrix} = \sigma_0^2 \begin{bmatrix} \dfrac{1}{n\sigma_y^2} & -\dfrac{\bar{y}}{n\sigma_y^2} \\ \bullet & \dfrac{1}{n} + \dfrac{\bar{y}^2}{n\sigma_y^2} \end{bmatrix}$$

interpolation: $y = ax + b$ and $x = cy + d$

interpolation variances: $\sigma_y^2 = x^2\sigma_a^2 + 2x\sigma_{ab} + \sigma_b^2 =$ $\sigma_x^2 = y^2\sigma_c^2 + 2y\sigma_{cd} + \sigma_d^2 =$

$$= \frac{\sigma_{0_v}^2}{n}\left(1 + \frac{(x-\bar{x})^2}{n\sigma_x^2}\right) \qquad = \frac{\sigma_{0_u}^2}{n}\left(1 + \frac{(y-\bar{y})^2}{n\sigma_y^2}\right)$$

residuals: $v = y_0 - y$ and $u = x_0 - x$

residual variances: $\sigma_v^2 = \sigma_{y_0}^2 - \sigma_y^2 =$ $\sigma_u^2 = \sigma_{x_0}^2 - \sigma_x^2 =$

$$= \sigma_{0_v}^2\left(\left(1 - \frac{1}{n}\right) - \frac{(x-\bar{x})^2}{n\sigma_x^2}\right) \qquad = \sigma_{0_u}^2\left(\left(1 - \frac{1}{n}\right) - \frac{(y-\bar{y})^2}{n\sigma_y^2}\right)$$

sigma naught: $\sigma_{0_v}^2 = \dfrac{1}{n-2}\sum_{i=1}^{n} v_i^2$ and $\sigma_{0_u}^2 = \dfrac{1}{n-2}\sum_{i=1}^{n} u_i^2$

linear correlation coefficient: $r_{xy} = \dfrac{\sigma_{xy}}{\sigma_x \sigma_y}$ (note: σ_x^2, σ_y^2 and σ_{xy} from data)

condition number: $\chi_\infty = n^2\sigma_x^2$ and $\chi_\infty = n^2\sigma_y^2$

local redundancy: $\nu_v = \left(1 - \dfrac{1}{n}\right) - \dfrac{(x-\bar{x})^2}{n\sigma_x^2}$ $\nu_u = \left(1 - \dfrac{1}{n}\right) - \dfrac{(y-\bar{y})^2}{n\sigma_y^2}$

[10] In the normal case.

robust regressions: $a = med(coef.ang._{ij}\ \forall\, i, j > i)$ and $c = med(coef.ang._{ij}\ \forall\, i, j > i)$

$b = med(y) - a\, med(x)$ $\quad d = med(x) - c\, med(y)$

interpolation: $y = ax + b$ $\quad x = cy + d$

residuals: $v = y_0 - y$ $\quad u = x_0 - x$

median abs. values: $mav(v) = med(|v|)$ $\quad mav(u) = med(|u|)$

rank correlation coefficient:

$$r_{xy} = 1 - \frac{6}{n(n^2-1)} \sum_{i=1}^{n} \left(rank_y - rank_x\right)_i^2$$

(note: $rank_x$ and $rank_y$ from data)

elementary transformations in the two-dimensional case:

- sum and difference: $s(o\,d) = x \pm y$ $\quad \bar{s}(o\,\bar{d}) = \bar{x} \pm \bar{y}$

  $$\sigma^2_{s(o\,d)} = \sigma^2_x + \sigma^2_y \pm 2\sigma_{xy}$$

- product: $p = x\,y$ $\quad \bar{p} = \bar{x}\,\bar{y}$

  $$\sigma^2_p = \frac{p^2}{x^2}\sigma^2_x + \frac{p^2}{y^2}\sigma^2_y + 2\frac{p^2}{xy}\sigma_{xy}$$

- quotient: $q = \dfrac{x}{y}$ $\quad \bar{q} = \dfrac{\bar{x}}{\bar{y}}$

  $$\sigma^2_q = \frac{q^2}{x^2}\sigma^2_x + \frac{q^2}{y^2}\sigma^2_y - 2\frac{q^2}{xy}\sigma_{xy}$$

Co-regression

- functional model: $\begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$ where: $dim(y_1) = m_1$ and $dim(y_2) = m_2$

  $dim(x_1) = x_1$ $\quad dim(x_2) = n_2$

- stochastic model: $\sigma_0^2 = \hat{\sigma}_0^2$ where: $\hat{\sigma}_0^2$ is coming from a separate regression

  $$R_{yy} = \begin{vmatrix} I & D \\ D & I \end{vmatrix}$$

  $d_{i(i+m)}$ is the correlation between two var.'s

  so that:

  $$R_{yy}^{-1} = \begin{vmatrix} \left(I - D^2\right)^{-1} & -D\left(I - D^2\right)^{-1} \\ -D\left(I - D^2\right)^{-1} & \left(I - D^2\right)^{-1} \end{vmatrix}$$

- least squares method: $\frac{1}{2}(\hat{y} - y_0)^T Q_{yy}^{-1}(\hat{y} - y_0) + \lambda^T(A\hat{x} - \hat{y}) = min$

- robust procedures: $\frac{1}{2}(\hat{y} - y_0)^T \psi\left(Q_{yy}^{-1}\right)(\hat{y} - y_0) + \lambda^T(A\hat{x} - \hat{y}) = min$

Least squares with equalities and inequalities

- Mathematical model for the quadratic programming:

$v = Ax + b$ with $x \geq 0$ and $v \geq 0$

being $v^T x = 0$

(note: $b = \delta - y^0$ and $v^T x = 0$ is equivalent to $\|v\|_2 = min$, i.e. least squares method)

- Linear complementary model, given by Karus–Kuhn–Tucker conditions and solvable by the Lemke-Ravindran algorithm:

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} A^T A & A \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} A^T b \\ b \end{bmatrix}$$

i.e.:

$w = Cz + d$ with $z \geq 0$ and $w \geq 0$

being $w^T z = 0$

- Mathematical model for linear la programming, solvable by Cottle-Dantzig algorithm:

$v = Ax + b$ with $x \geq 0$ and $v \geq 0$

being $\|v\|_1 = min$

Wavelet analysis

Note: recalling of the general expression for interpolation in one- and two-dimensional cases:

$$y_k = \sum_i a_i^{(l)} F_i \left(\left(x_k - x_l^{min} \right) / \left(x_l^{max} - x_l^{min} \right) \right)$$ with i = I, II, III, …

$$z_k = \sum_i \sum_j a_{ij}^{(l)} F_{ij} \left(\left(x_k - x_l^{min} \right) / \left(x_l^{max} - x_l^{min} \right), \left(y_k - y_l^{min} \right) / \left(y_l^{max} - y_l^{min} \right) \right)$$

with i = I, II, III, … and j = 1, 2, 3, …

multi-dimensional cases are performed applying the well-known associative property of the product, both in construction of the basic functions and in implementation of the interpolation.

Wavelet interpolation (in the frequency domain):

$$\psi_{(a,b)}(t) = \frac{1}{a} \psi \left(\frac{t-b}{a} \right) \qquad a > 0$$

Examples of *Wavelet* mothers:

- Haar *Wavelet*:

$$\psi(t) = \begin{cases} 1 & 0 \le t \le 1/2 \\ -1 & 1/2 \le t \le 1 \\ 0 & otherwise \end{cases}$$

- Shannon (or *sinc*) *Wavelet*:

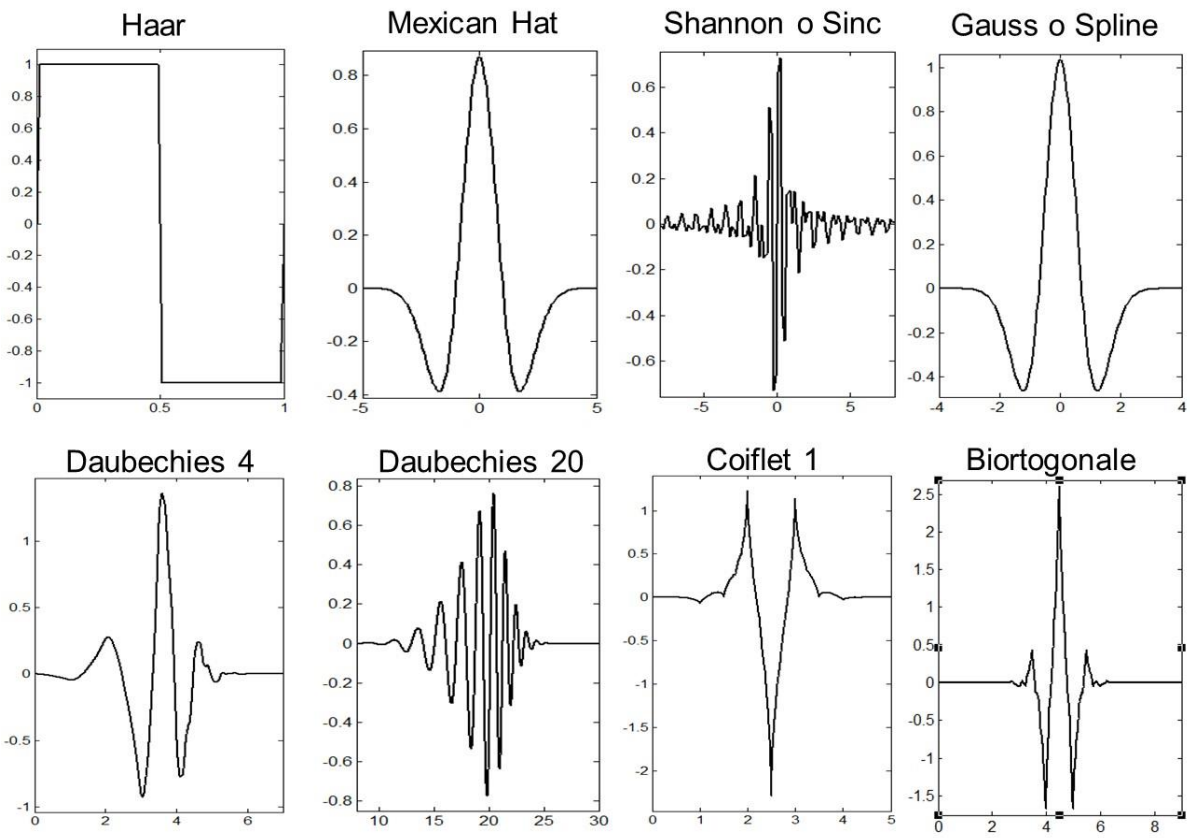
$$\psi(t) = sin\, c\, \frac{t}{2} \cdot cos\, \frac{3\pi\, t}{2}$$

- Ricker or Mexican Hat *Wavelet*:

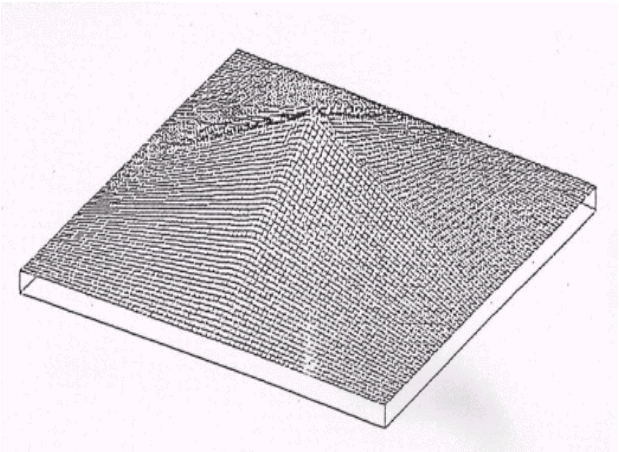
$$\psi(t) = \frac{2}{\sqrt{3\pi^{1/4}\sigma}} \left(1 - \frac{t^2}{4}\right) \cdot e^{-\frac{t^2}{2\sigma^2}}$$

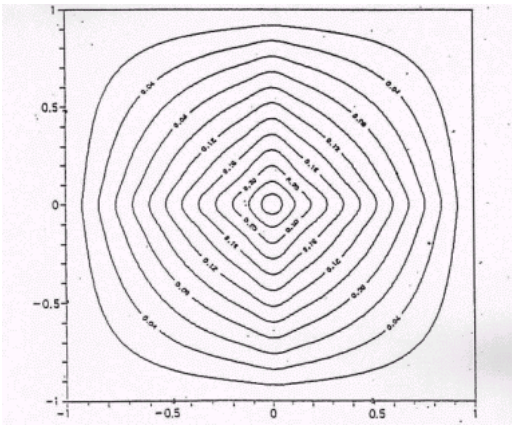
- Quadratic B-spline *Wavelet*:

$$\psi(t) = \begin{cases} \frac{t^2}{2} & 0 \le t < 1 \\ -t^2 + 3t - \frac{3}{2} & 1 \le t < 2 \\ \frac{t^2}{2} - 3t + \frac{9}{2} & 2 \le t < 3 \\ 0 & otherwise \end{cases}$$

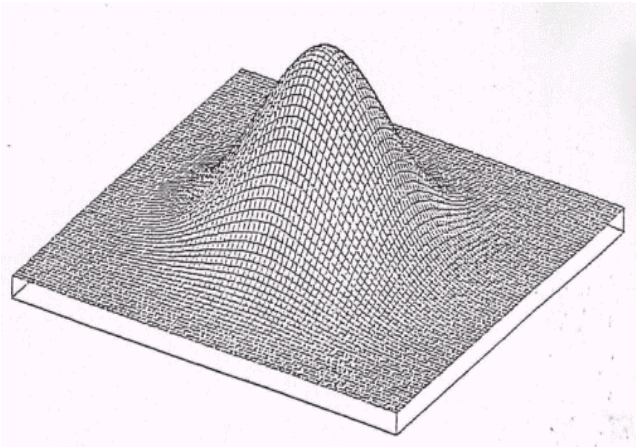


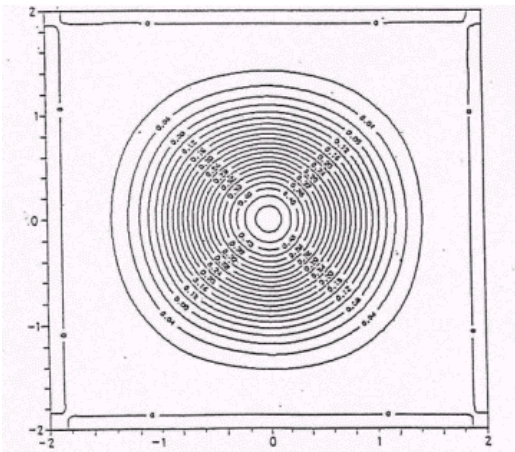
Wavelet mother examples





Axonometric picture and contour line map of bi-linear function





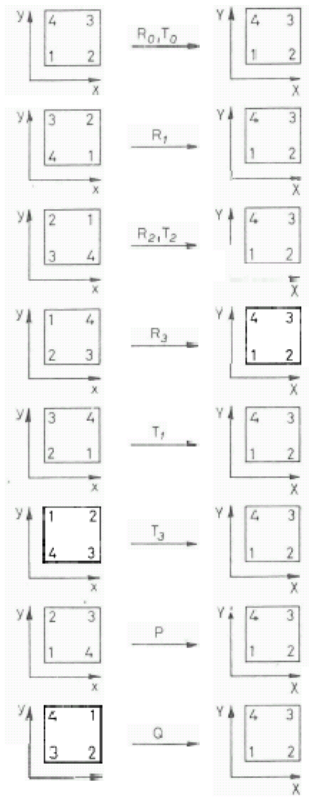
Axonometric picture and contour line map of bi-cubic function

Topological transformations and symmetry groups

<u>S4 permutation group</u> (with 4!=24 elements, where its rigid subgroup has 8 elements)

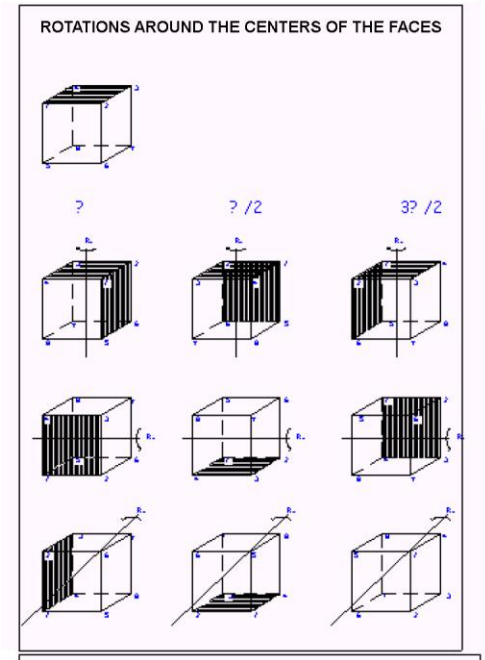
1.	1	2	3	4	identity
2.	1	3	4	2	triple exchanges (not rigid)
3.	1	4	2	3	
4.	1	4	3	2	exchange: glisso–reflection on the bisector of the 1st e 3rd quadrant (P)
5.	1	3	2	4	exchanges (not rigid)
6.	1	2	4	3	
7.	2	3	4	1	simple counterclockwise rotation (R_1)
8.	2	3	1	4	triple exchanges (not rigid)
9.	2	4	3	1	
10.	2	1	4	3	double exchange: reflection on the vertical axis (T_1)
11.	2	1	3	4	exchange (not rigid)
12.	2	4	1	3	quadruple exchange (not rigid)
13.	3	4	1	2	double counterclockwise rotation, double exchange: double reflection (R_2, T_2)
14.	3	1	2	4	triple exchanges (not rigid)
15.	3	2	4	1	
16.	3	2	1	4	exchange: glisso–reflection on the bisector of the 2nd e 4th quadrant (Q)
17.	3	1	4	2	quadruple exchanges (not rigid)
18.	3	4	2	1	
19.	4	1	2	3	triple counterclockwise rotation (R_3)
20.	4	1	3	2	triple exchanges (not rigid)
21.	4	2	1	3	
22.	4	3	2	1	double exchange: reflection on the horizontal axis (T_3)
23.	4	2	3	1	exchange (not rigid)
24.	4	3	1	2	quadruple exchange (not rigid)

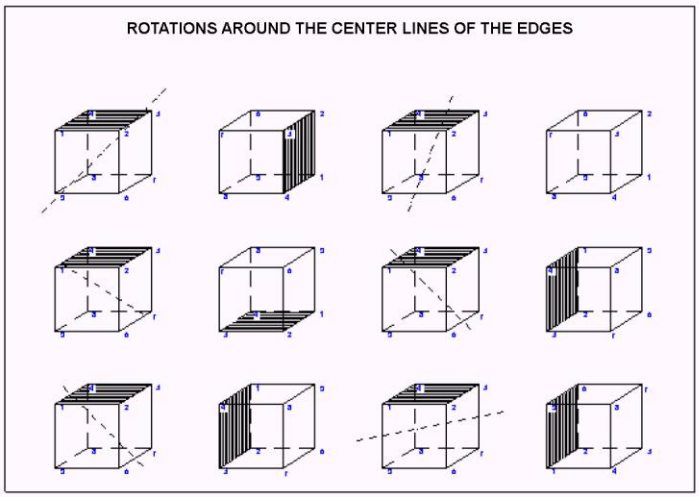
1. R_0, T_0 $\begin{cases} X \equiv x \\ Y \equiv y \end{cases}$
2. R_1 $\begin{cases} X \equiv y \\ Y \equiv -x \end{cases}$
3. R_2, T_2 $\begin{cases} X \equiv -x \\ Y \equiv -y \end{cases}$
4. R_3 $\begin{cases} X \equiv -y \\ Y \equiv x \end{cases}$
5. T_1 $\begin{cases} X \equiv -x \\ Y \equiv y \end{cases}$
6. T_3 $\begin{cases} X \equiv x \\ Y \equiv -y \end{cases}$
7. P $\begin{cases} X \equiv y \\ Y \equiv x \end{cases}$
8. Q $\begin{cases} X \equiv -y \\ Y \equiv -x \end{cases}$

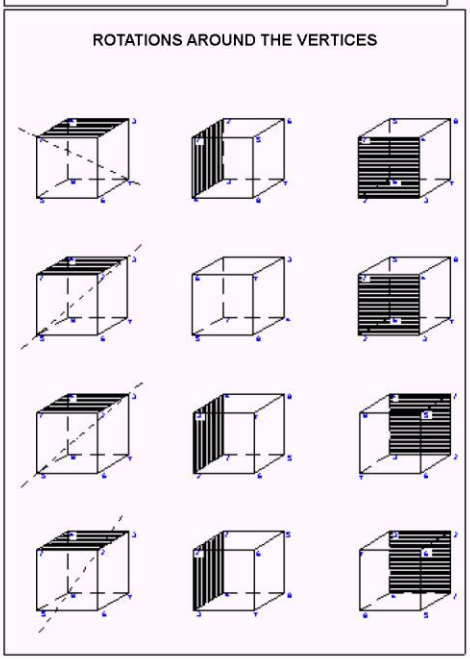


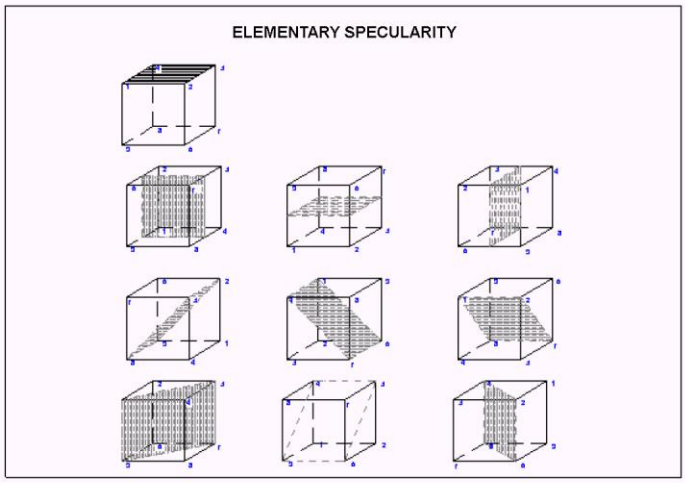
S8 permutation group (with 8!=40320 elements, but its rigid subgroup has 48 elements only)

1.	1	2	3	4	5	6	7	8	identity
2.	4	1	2	3	8	5	6	7	rotations around the *z* axis
3.	3	4	1	2	7	8	5	6	
4.	2	3	4	1	6	7	8	5	
5.	4	3	7	8	1	2	6	5	rotations around the *x* axis
6.	8	7	6	5	4	3	2	1	
7.	5	6	2	1	8	7	3	4	
8.	2	6	7	3	1	5	8	4	rotations around the *y* axis
9.	6	5	8	7	2	1	4	3	
10.	5	1	4	8	6	2	3	7	
11.	1	5	6	2	4	8	7	3	rotations around the vertices
12.	1	4	8	5	2	3	7	6	
13.	6	7	3	2	5	8	4	1	
14.	8	4	3	7	5	1	2	6	
15.	3	2	6	7	4	1	5	8	
16.	6	2	1	5	7	3	4	8	
17.	3	7	8	4	2	6	5	1	
18.	8	5	1	4	7	6	2	3	
19.	7	3	2	6	8	4	1	5	rotations around the middle point of the edges
20.	4	8	5	1	3	7	6	2	
21.	7	8	4	3	6	5	1	2	
22.	2	1	5	6	3	4	8	7	
23.	7	6	5	8	3	2	1	4	
24.	5	8	7	6	1	4	3	2	
25.	5	6	7	8	1	2	3	4	elementary specularities
26.	2	1	4	3	6	5	8	7	
27.	4	3	2	1	8	7	6	5	
28.	3	2	1	4	7	6	5	8	
29.	1	4	3	2	5	8	7	6	
30.	6	2	3	7	5	1	4	8	
31.	1	5	8	4	2	6	7	3	
32.	1	2	5	6	4	3	7	8	
33.	8	7	3	4	5	6	2	1	composed specularities
34.	8	5	6	7	4	1	2	3	
35.	7	8	5	6	3	4	1	2	
36.	6	7	8	5	2	3	4	1	
37.	8	7	3	4	5	6	2	1	
38.	4	3	2	1	8	7	6	5	
39.	1	2	6	5	4	3	7	8	
40.	6	2	3	7	5	1	4	8	
41.	2	1	4	3	6	5	8	7	
42.	1	5	8	4	2	6	7	3	
43.	3	7	6	2	4	8	1	5	
44.	8	4	1	5	7	3	2	6	
45.	3	4	8	7	2	1	5	6	
46.	6	5	1	2	7	8	4	3	
47.	3	2	1	4	7	6	5	8	
48.	1	4	3	2	5	8	7	6	

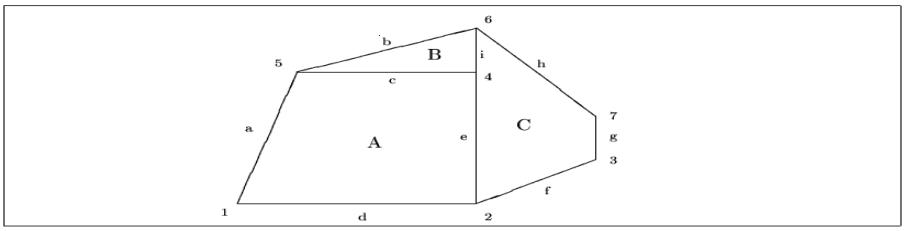


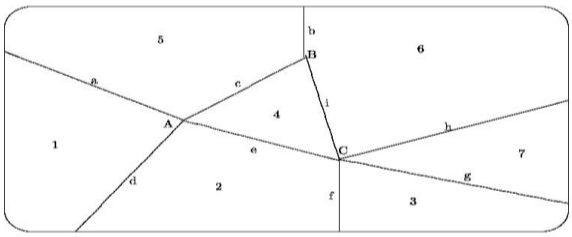






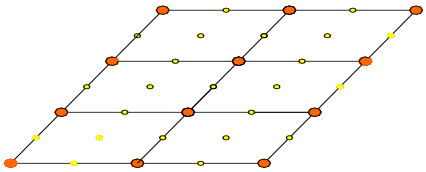
<u>Original and dual 2D graphs</u> (exchanging points and regions)



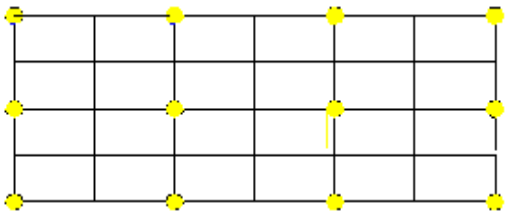


2D lattices

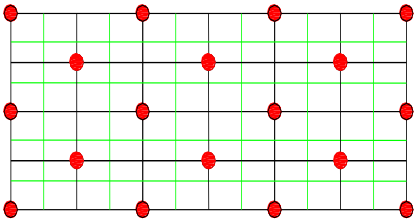
1. Parallelogram lattice (translations and 180° rotations)



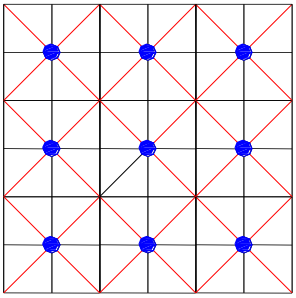
2. Rectangular lattice (translations, 180° rotations and reflections)



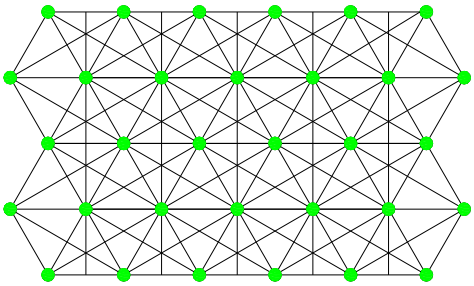
3. Rhombic lattice (translations, 180° rotations, reflections and glisso-reflections)



4. Square lattice (translations, 90° and 180° rotations, reflections and glisso-reflections)



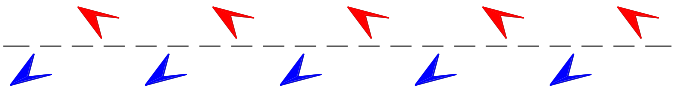
5. Hexagonal lattice (translations, 60°, 120° and 180° rotations, reflections and glisso-reflections)



Linear symmetry

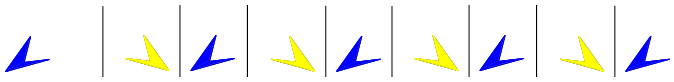
- **p111** translations: F F F F F F (1)

- **p1a1** translations and glisso-reflections: p b p b p b (2)

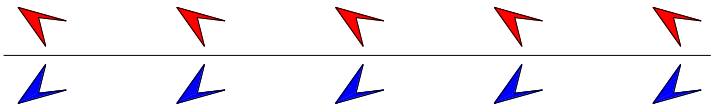


- **p112** translations and 180°rotations: S S S S S S (3)

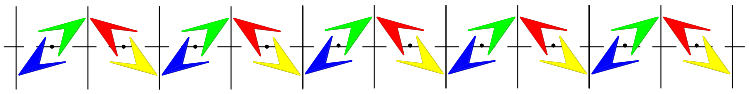
- **pm11** translations and orthogonal reflections: A A A A A A (4)



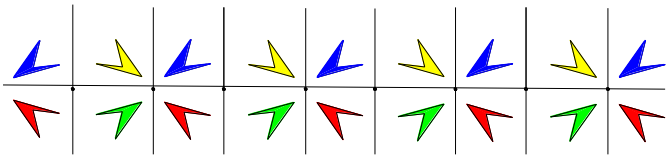
- **p1m1** translations and parallel reflections: E E E E E E (5)



- **pma2** translations, 180° rotations, orthogonal reflections and glisso-reflections:
  bd pq bd pq bd pq (6)



- **pmm2** translations, 90° and 180° rotations, parallel and orthogonal reflections:
  X X X X X X (7)

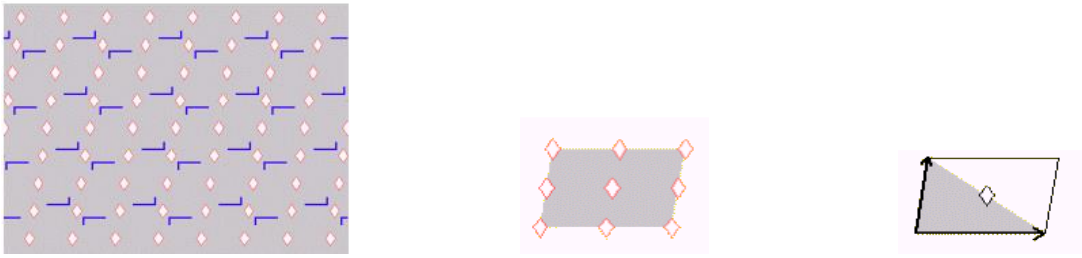


Plane symmetry

- **p111 = p1** translations (parallelogram lattice with parallelogram fundamental region)

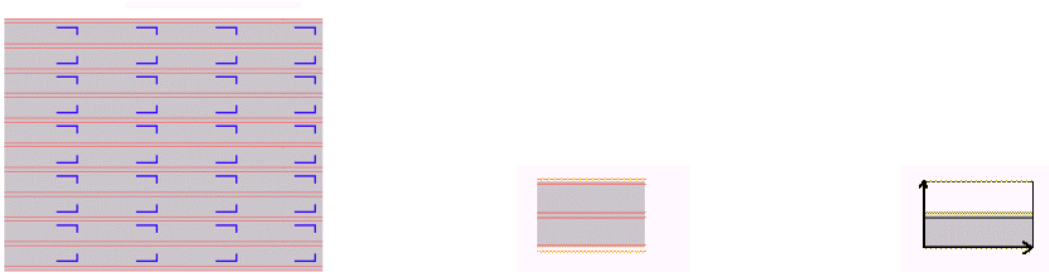
(1)

- **p211 = p2** translations and 180° rotations (parallelogram lattice with triangular fundamental region)



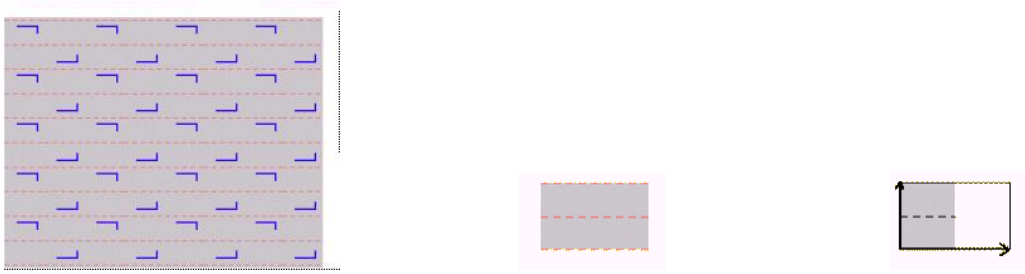
(2)

- **p1m1 = pm** translations and reflections (rectangular lattice with rectangular fundamental region)



(3)

- **p1g1 = pg** translations and glisso-reflections (rectangular lattice with rectangular fundamental region)

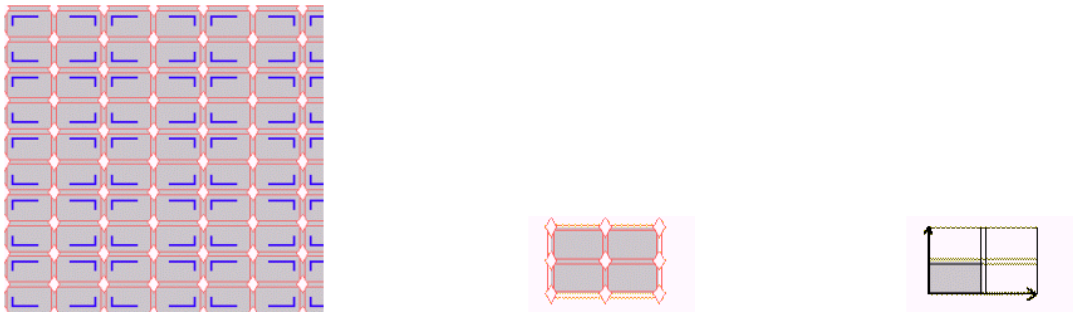


(4)

- **c1m1 = cm** translations, reflections and glisso-reflections (rhombic lattice with triangular fundamental region)

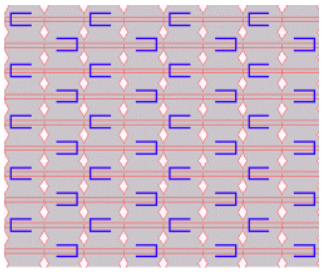
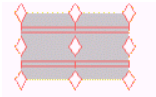
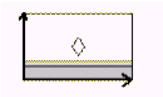
(5)

- **p2mm = pmm** translations, reflections and 180° rotations (rectangular lattice with rectangular fundamental region)



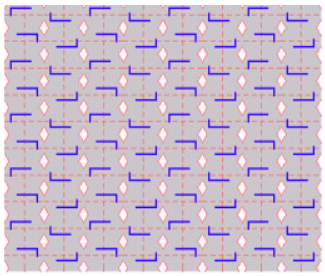
(6)

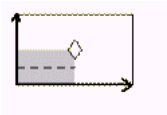
- **p2mg = pmg** translations, reflections, glisso-reflections and 180° rotations (rectangular lattice with two different rectangular fundamental regions)

(7)

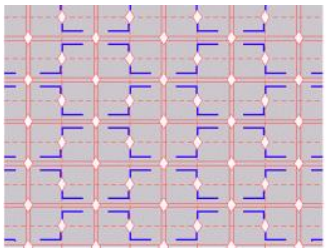
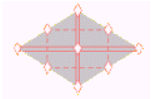
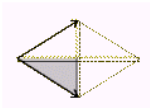
- **p2gg = pgg** translations, glisso-reflections and 180° rotations (rectangular lattice with two different rectangular fundamental regions)





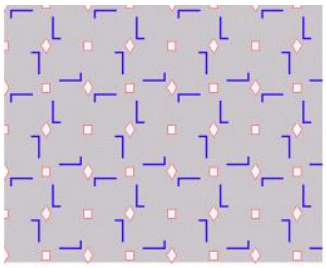
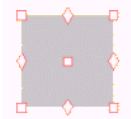
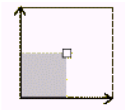
(8)

- **c2mm = cmm** translations, reflections, glisso-reflections and 180° rotations (rhombic lattice with two different rhombic and triangular fundamental regions)

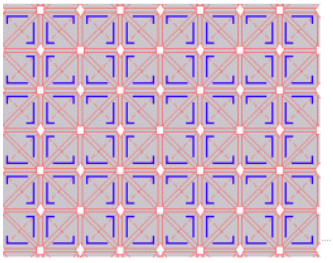
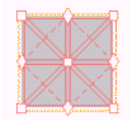
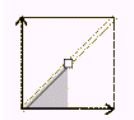
(9)

- **p411 = p4** translations, 90° and 180° rotations (square lattice with two different square fundamental regions)

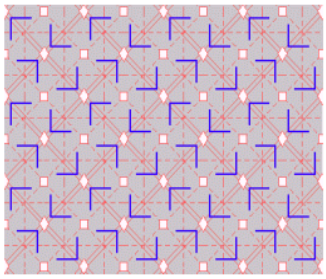
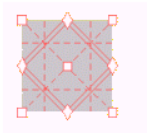
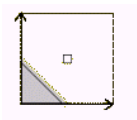
(10)

- **p4mm = p4m** translations, reflections, glisso-reflections, 90° and 180° rotations (square lattice with triangular fundamental region)

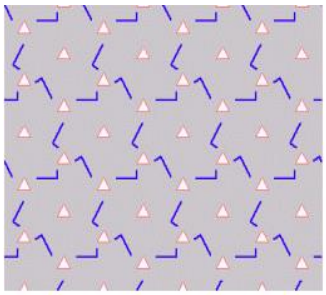
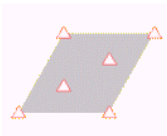
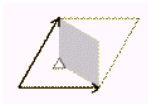
(11)

- **p4gm = p4g** translations, reflections, glisso-reflections, 90° and 180° rotations (square lattice with triangular fundamental region)

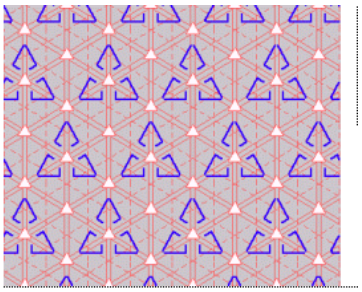
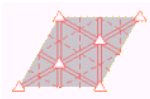
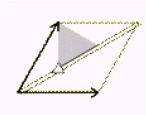
(12)

- **p311 = p3** translation and 120° rotations (rhombic lattice with rhombic fundamental region)

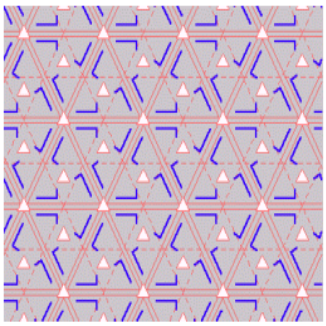
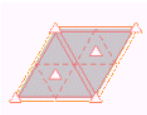
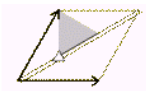
(13)

- **p31m = p31m** translations, reflections, glisso-reflections and 120° rotations (rhombic lattice with triangular fundamental region)

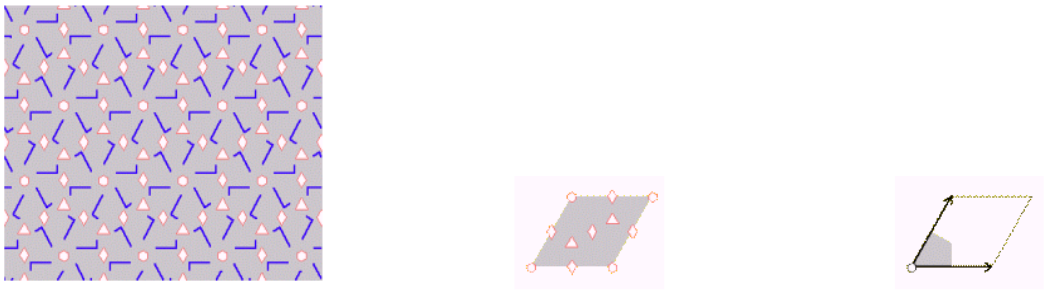
(14)

- **p3m1 = p3m1** translations, reflections, glisso-reflections and 120° rotations (rhombic lattice with triangular fundamental region)

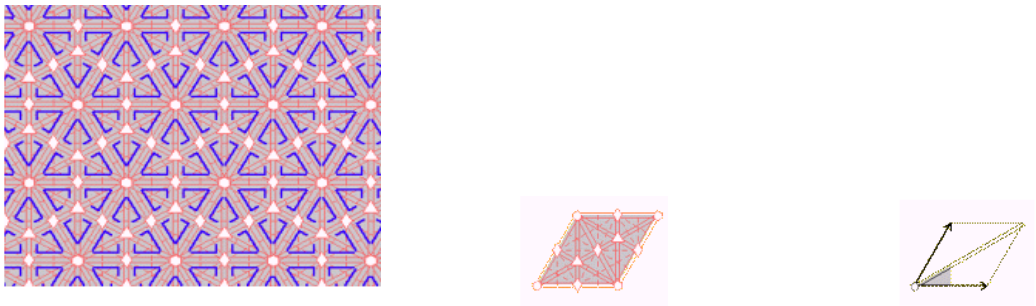
(15)

- **p611 = p6** translations, 60°, 120° and 180° rotations (rhombic lattice with “kite” fundamental region)

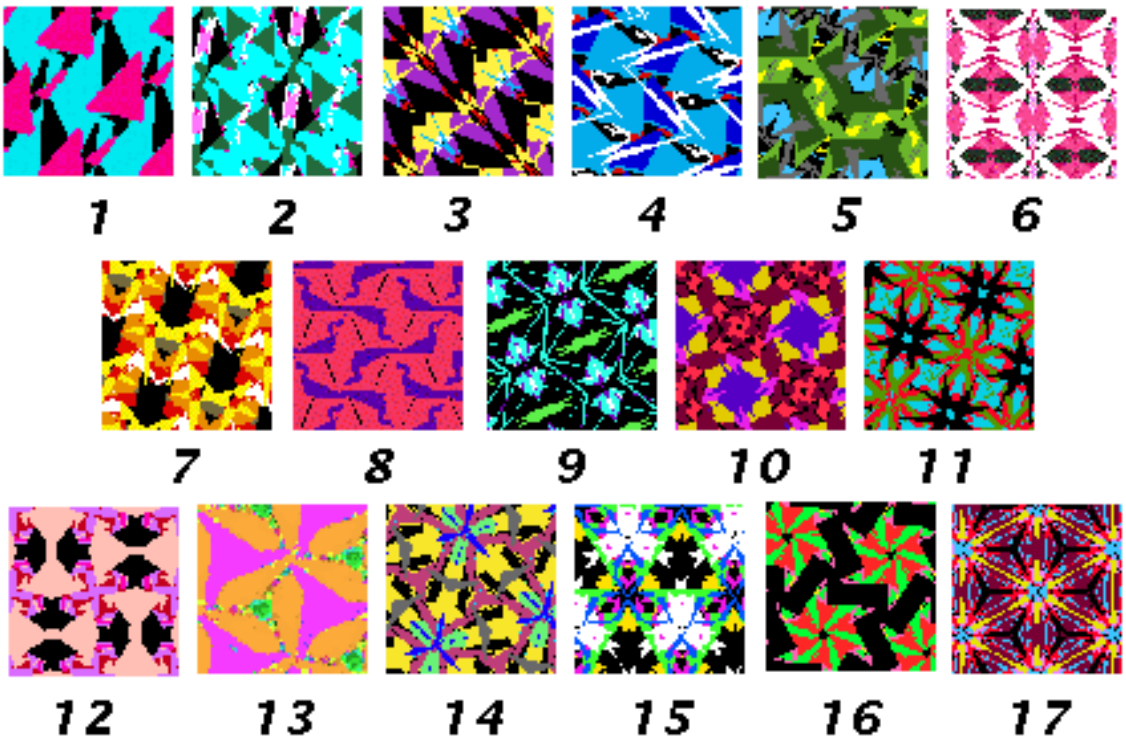


(16)

- **p6mm = p6m** translations, reflection, glisso-reflections, 60°, 120° and 180° rotations (rhombic lattice with triangular fundamental region)



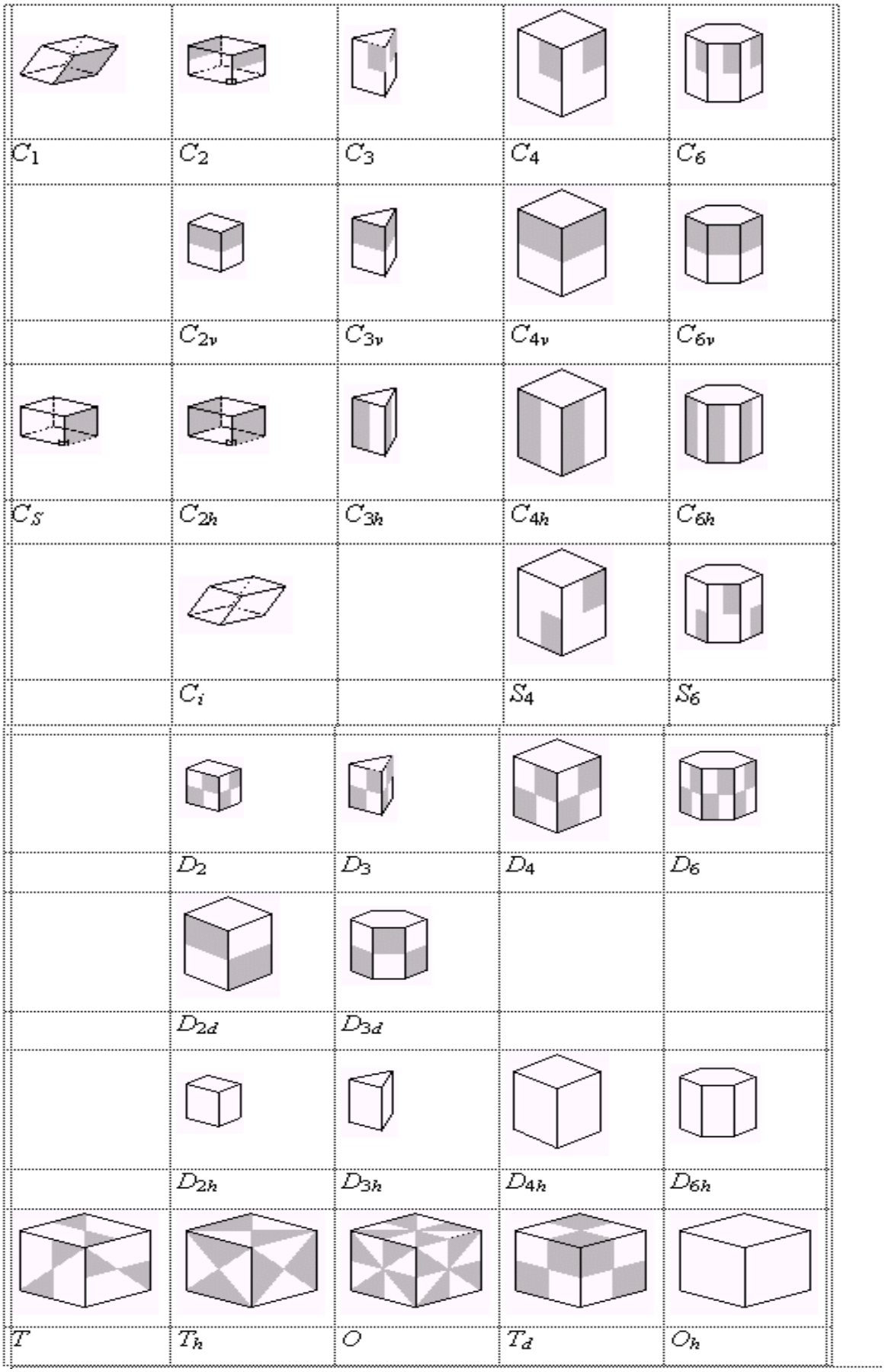
(17)



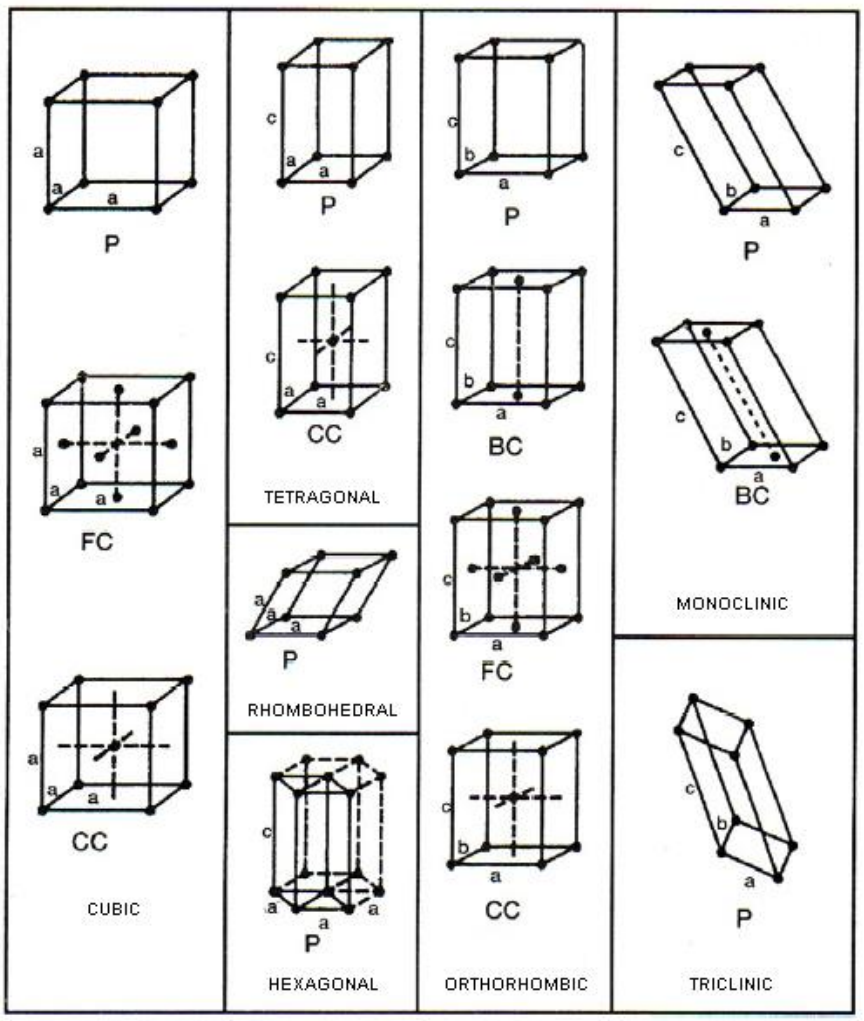
Plane tessellation

<u>Spatial symmetry</u>

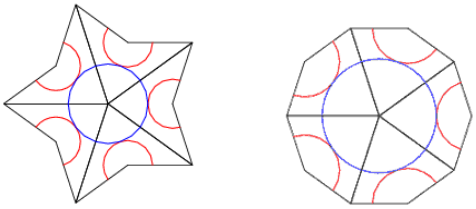
1. triclinic
2. monoclinic
3. orthorhombic
4. trigonal
5. tetragonal
6. hexagonal
7. cubic



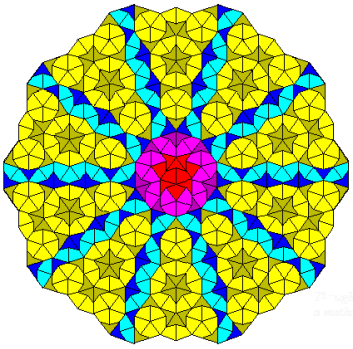
32 crystal groups of symmetry



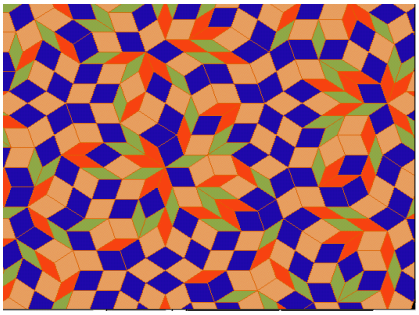
14 Bravais' spatial lattices (for a total number of 230 3D group of symmetry)



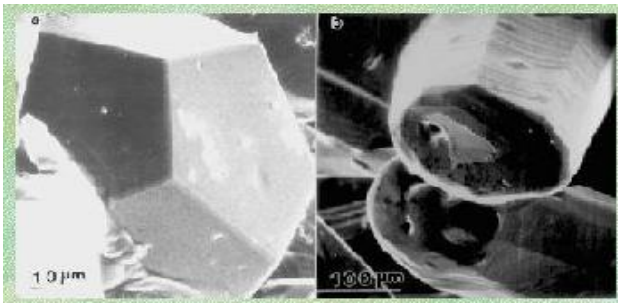
Penrose's star and sun



Penrose's a-periodic tessellation



3D a-periodic tessellation



Quasi-crystals

PART II – PHOTOGRAMMETRY

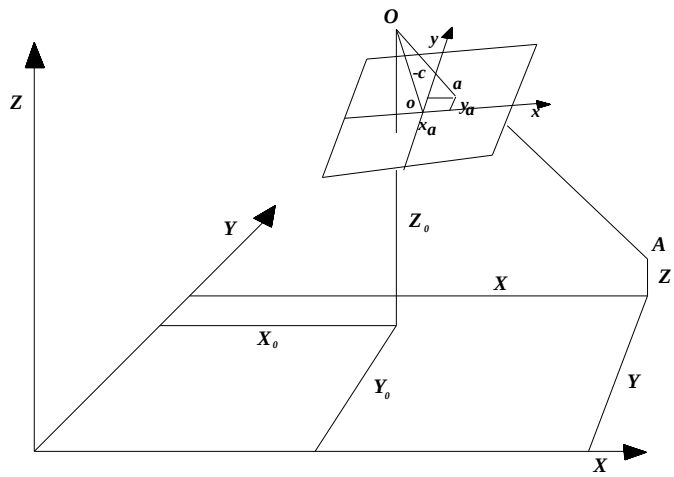
From image space to object space

- 3D S-transformation

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix} = \lambda R \left(\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} - \begin{vmatrix} X_0 \\ Y_0 \\ Z_0 \end{vmatrix} \right)$$

Legend:

- (x, y, z) image coordinates of the image point Q
- (X, Y, Z) image coordinates of the object point P
- (X_o, Y_o, Z_o) coordinates of the projection center
- λ variation scale
- $R = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix}$ rotation matrix, where: $\sum_i \hat{r}_{ki}^2 = 1 \quad k = 1, 2, 3$; $\sum_i \hat{r}_{ki}\hat{r}_{li} = 0 \quad k, l = 1\text{-}2, 1-3, 2-3$



- 3D S-transformation versus collinearity equations:

$$\begin{vmatrix} x^\circ \\ y^\circ \\ -c \end{vmatrix}_{ij} = \hat{\lambda}_{ij} \hat{R}_j \left(\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i - \begin{vmatrix} \hat{X}_0 \\ \hat{Y}_0 \\ \hat{Z}_0 \end{vmatrix}_j \right)$$

being: $z_{ij} = const = c \qquad \forall ij$

Rotation matrices

<u>Linking between Euler's angles and the angles traditionally used in surveying and geodesy:</u>

- surveying:

$\vartheta = 2\pi - \vartheta$ $\qquad$ ϑ azimuthal angle

$\zeta = \pi - \zeta$ $\qquad$ ζ zenithal angle

$\alpha = \alpha$ $\qquad$ $\alpha \equiv 0$

- geodesy:

$\vartheta = \pi/2 \mp \lambda$ $\qquad$ longitude

$\zeta = \pi/2 \mp \varphi$ $\qquad$ geocentric latitude $\qquad$ (note: $\zeta = \pi/2 \mp arctan\left[\left(1-e^2\right)tan\left|\varphi\right|\right]$)

$\alpha = \pi/2 - \alpha$ $\qquad$ azimuth

<u>Rodriguez matrix:</u>

$$R = (I-S)(I+S)^{-1} \qquad \text{where:} \qquad S = \frac{1}{2}\begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$

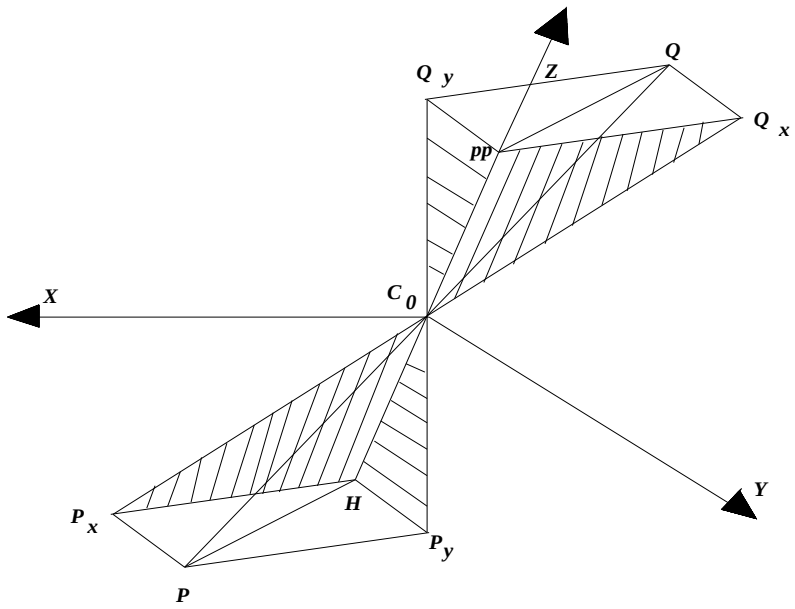
$$I - S = \begin{vmatrix} 1 & -c/2 & b/2 \\ c/2 & 1 & -a/2 \\ -b/2 & a/2 & 1 \end{vmatrix} \qquad I + S = \begin{vmatrix} 1 & c/2 & -b/2 \\ -c/2 & 1 & a/2 \\ b/2 & -a/2 & 1 \end{vmatrix}$$

$$det(I+S) = 1 + \frac{a^2+b^2+c^2}{4} \qquad (I+S)^{-1} = \frac{1}{1+\frac{a^2+b^2+c^2}{4}}\begin{vmatrix} 4+a^2 & ab-2c & ac+2b \\ ab+2c & 4+b^2 & bc-2a \\ ac-2b & bc+2a & 4+c^2 \end{vmatrix}$$

$$R = \frac{1}{1+\frac{a^2+b^2+c^2}{4}}\begin{vmatrix} 1+\frac{\left(a^2-b^2-c^2\right)}{4} & -c+\frac{1}{2}ab & b+\frac{1}{2}ac \\ c+\frac{1}{2}ab & 1+\frac{\left(-a^2+b^2-c^2\right)}{4} & -a+\frac{1}{2}bc \\ -b+\frac{1}{2}ac & a+\frac{1}{2}bc & 1+\frac{\left(-a^2-b^2+c^2\right)}{4} \end{vmatrix}$$

(note: first order development: $\qquad \tilde{R} = \begin{vmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{vmatrix} = \begin{vmatrix} 1 & k & -\varphi \\ -k & 1 & \omega \\ \varphi & -\omega & 1 \end{vmatrix}$

Collinearity equations



<u>3D Thales' theorem:</u>

$$\frac{PPQ_x}{PPC} = \frac{HP_x}{HC}$$

$$\frac{PPQ_y}{PPC} = \frac{HP_y}{HC}$$

- using the internal direction tangents:

$$\frac{x^\circ}{-c} = \frac{\hat{r}_{11}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{12}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{13}\left(\hat{Z}-\hat{Z}_0\right)}{\hat{r}_{31}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{32}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{33}\left(\hat{Z}-\hat{Z}_0\right)}$$

$$\frac{y^\circ}{-c} = \frac{\hat{r}_{21}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{22}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{23}\left(\hat{Z}-\hat{Z}_0\right)}{\hat{r}_{31}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{32}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{33}\left(\hat{Z}-\hat{Z}_0\right)}$$

- using the image coordinates:

$$x^\circ = -c\frac{\hat{r}_{11}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{12}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{13}\left(\hat{Z}-\hat{Z}_0\right)}{\hat{r}_{31}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{32}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{33}\left(\hat{Z}-\hat{Z}_0\right)}$$

$$y^\circ = -c\frac{\hat{r}_{21}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{22}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{23}\left(\hat{Z}-\hat{Z}_0\right)}{\hat{r}_{31}\left(\hat{X}-\hat{X}_0\right)+\hat{r}_{32}\left(\hat{Y}-\hat{Y}_0\right)+\hat{r}_{33}\left(\hat{Z}-\hat{Z}_0\right)}$$

- linearization of the second expressions by a first order Taylor's expansions:

$$x = \tilde{x} + \left(\frac{\partial x}{\partial X_0}\right)\delta X_0 + \left(\frac{\partial x}{\partial Y_0}\right)\delta Y_0 + \left(\frac{\partial x}{\partial Z_0}\right)\delta Z_0 + \left(\frac{\partial x}{\partial \omega}\right)\delta\omega + \left(\frac{\partial x}{\partial \varphi}\right)\delta\varphi + \left(\frac{\partial x}{\partial k}\right)\delta k +$$
$$+\left(\frac{\partial x}{\partial X}\right)\delta X + \left(\frac{\partial x}{\partial Y}\right)\delta Y + \left(\frac{\partial x}{\partial Z}\right)\delta Z$$

$$y = \tilde{y} + \left(\frac{\partial y}{\partial X_0}\right)\delta X_0 + \left(\frac{\partial y}{\partial Y_0}\right)\delta Y_0 + \left(\frac{\partial y}{\partial Z_0}\right)\delta Z_0 + \left(\frac{\partial y}{\partial \omega}\right)\delta\omega + \left(\frac{\partial y}{\partial \varphi}\right)\delta\varphi + \left(\frac{\partial y}{\partial k}\right)\delta k +$$
$$+ \left(\frac{\partial y}{\partial X}\right)\delta X + \left(\frac{\partial y}{\partial Y}\right)\delta Y + \left(\frac{\partial y}{\partial Z}\right)\delta Z$$

where:

$$\frac{\partial x}{\partial X_0} = \frac{1}{D}\left(cr_{11} + xr_{31}\right) = \frac{1}{D}\left(c\cos\varphi\cos k + x\sin\varphi\right)$$

$$\frac{\partial x}{\partial Y_0} = \frac{1}{D}\left(cr_{12} + xr_{32}\right) = \frac{1}{D}\left[c\left(\cos\omega\sin k + \sin\omega\sin\varphi\cos k\right) - x\sin\omega\cos\varphi\right]$$

$$\frac{\partial x}{\partial Z_0} = \frac{1}{D}\left(cr_{13} + xr_{33}\right) = \frac{1}{D}\left[c\left(\sin\omega\sin k - \cos\omega\sin\varphi\cos k\right) + x\cos\omega\cos\varphi\right]$$

$$\frac{\partial x}{\partial \omega} = -\frac{1}{D}\left[\left(-cr_{13} + xr_{33}\right)\left(Y - Y_0\right) + \left(cr_{12} + xr_{32}\right)\left(Z - Z_0\right)\right] =$$
$$= -\frac{1}{D}\left\{\left[c\left(-\sin\omega\sin k + \cos\omega\sin\varphi\cos k\right) + x\cos\omega\cos\varphi\right]\left(Y - Y_0\right) +\right.$$
$$\left. + \left[c\left(\cos\omega\sin k + \sin\omega\sin\varphi\cos k\right) - x\sin\omega\cos\varphi\right]\left(Z - Z_0\right)\right\}$$

$$\frac{\partial x}{\partial \varphi} = -\frac{1}{D}\left[\left(-c\sin\varphi\cos k + x\cos\varphi\right)\left(X - X_0\right) + \left(c\sin\omega\cos\varphi\cos k + x\sin\omega\sin\varphi\right)\left(Y - Y_0\right) +\right.$$
$$\left. - \left(c\cos\omega\cos\varphi\cos k + x\cos\omega\sin\varphi\right)\left(Z - Z_0\right)\right]$$

$$\frac{\partial x}{\partial k} = y = -\frac{c}{D}\left[-\cos\varphi\sin k\left(X - X_0\right) + \left(c\cos\omega\cos k + \sin\omega\sin\varphi\sin k\right)\left(Y - Y_0\right) +\right.$$
$$\left. + \left(\sin\omega\cos k + \cos\omega\sin\varphi\sin k\right)\left(Z - Z_0\right)\right]$$

$$\frac{\partial y}{\partial X_0} = \frac{1}{D}\left(cr_{21} + yr_{31}\right) = \frac{1}{D}\left(-c\cos\varphi\sin k + y\sin\varphi\right)$$

$$\frac{\partial y}{\partial Y_0} = \frac{1}{D}\left(cr_{22} + yr_{32}\right) = \frac{1}{D}\left[c\left(\cos\omega\cos k - \sin\omega\sin\varphi\sin k\right) - y\sin\omega\cos\varphi\right]$$

$$\frac{\partial y}{\partial Z_0} = \frac{1}{D}\left(cr_{13} + yr_{33}\right) = \frac{1}{D}\left[c\left(\sin\omega\cos k + \cos\omega\sin\varphi\sin k\right) + y\cos\omega\cos\varphi\right]$$

$$\frac{\partial y}{\partial \omega} = -\frac{1}{D}\left[-\left(cr_{23} + yr_{33}\right)\left(Y - Y_0\right) + \left(cr_{22} + yr_{32}\right)\left(Z - Z_0\right)\right] =$$
$$= -\frac{1}{D}\left\{-\left[c\left(\sin\omega\cos k + \cos\omega\sin\varphi\sin k\right) + y\cos\omega\cos\varphi\right]\left(Y - Y_0\right) +\right.$$

$$+\left[c\left(\cos\omega\cos k-\sin\omega\sin\varphi\sin k\right)-y\sin\omega\cos\varphi\right]\left(Z-Z_0\right)\}$$

$$\frac{\partial y}{\partial\varphi}=-\frac{1}{D}\left[\left(c\sin\varphi\sin k+y\cos\varphi\right)\left(X-X_0\right)+\left(-c\sin\omega\cos\varphi\sin k+y\sin\omega\sin\varphi\right)\left(Y-Y_0\right)\right.$$

$$\left.+\left(c\cos\omega\cos\varphi\sin k-y\cos\omega\sin\varphi\right)\left(Z-Z_0\right)\right]$$

$$\frac{\partial y}{\partial k}=-x=\frac{c}{D}\left[\cos\varphi\cos k\left(X-X_0\right)+\left(\cos\omega\sin k+\sin\omega\sin\varphi\cos k\right)\left(Y-Y_0\right)+\right.$$

$$\left.+\left(\sin\omega\sin k-\cos\omega\sin\varphi\cos k\right)\left(Z-Z_0\right)\right]$$

moreover:

$$\frac{\partial x}{\partial X}=-\frac{\partial x}{\partial X_0};\quad \frac{\partial x}{\partial Y}=-\frac{\partial x}{\partial Y_0};\quad \frac{\partial x}{\partial Z}=-\frac{\partial x}{\partial Z_0}$$

$$\frac{\partial y}{\partial X}=-\frac{\partial y}{\partial X_0};\quad \frac{\partial y}{\partial Y}=-\frac{\partial x}{\partial Y_0};\quad \frac{\partial y}{\partial Z}=-\frac{\partial y}{\partial Z_0}$$

being: $D=r_{31}\left(X-X_0\right)+r_{32}\left(Y-Y_0\right)+r_{33}\left(Z-Z_0\right)$

<u>In geodetic coordinates:</u>

$$x=\lambda R\left(X-X_0\right)\qquad x=-c\frac{R_1\left(X-X_0\right)}{R_3\left(X-X_0\right)}$$

$$y=-c\frac{R_2\left(X-X_0\right)}{R_3\left(X-X_0\right)}$$

being: $G=\begin{vmatrix}\varphi\\ \lambda\\ h\end{vmatrix};\qquad G_0=\begin{vmatrix}\varphi_0\\ \lambda_0\\ h_0\end{vmatrix}\qquad X=X\left(G\right)=\begin{vmatrix}\left(N+h\right)\cos\varphi\cos\lambda\\ \left(N+h\right)\cos\varphi\sin\lambda\\ \left(N\left(1-e^2\right)+h\right)\sin\varphi\end{vmatrix}$

$$\psi=\begin{vmatrix}\omega\\ \varphi\\ k\end{vmatrix};\qquad \begin{matrix}r_1=r_1\left(\psi\right)\\ r_2=r_2\left(\psi\right)\\ r_3=r_3\left(\psi\right)\end{matrix}$$

(note: $h=H+N$ [11])

$$x=\tilde{x}+\frac{\partial x}{\partial X}\frac{\partial X}{\partial G}\delta G+\frac{\partial x}{\partial X_0}\frac{\partial X_0}{\partial G_0}\delta G_0+\left(\frac{\partial x}{\partial r_1}\frac{\partial r_1}{\partial\psi}+\frac{\partial x}{\partial r_3}\frac{\partial r_3}{\partial\psi}\right)\delta\psi$$

$$y=\tilde{y}+\frac{\partial y}{\partial X}\frac{\partial X}{\partial G}\delta G+\frac{\partial y}{\partial X_0}\frac{\partial X_0}{\partial G_0}\delta G_0+\left(\frac{\partial y}{\partial r_2}\frac{\partial r_2}{\partial\psi}+\frac{\partial y}{\partial r_3}\frac{\partial r_3}{\partial\psi}\right)\delta\psi$$

[11] N geoids undulation.

Splitting in two groups the external parameters of the collinearity equations of two images

$$x^\circ_1 = -c\frac{\hat{r}_{11.1}\left(\hat{X}_1-\hat{X}_{01}\right)+\hat{r}_{12.1}\left(\hat{Y}_1-\hat{Y}_{01}\right)+\hat{r}_{13.1}\left(\hat{Z}_1-\hat{Z}_{01}\right)}{\hat{r}_{31.1}\left(\hat{X}_1-\hat{X}_{01}\right)+\hat{r}_{32.1}\left(\hat{Y}_1-\hat{Y}_{01}\right)+\hat{r}_{33.1}\left(\hat{Z}_1-\hat{Z}_{01}\right)}$$

$$y^\circ_1 = -c\frac{\hat{r}_{21.1}\left(\hat{X}_1-\hat{X}_{01}\right)+\hat{r}_{22.1}\left(\hat{Y}_1-\hat{Y}_{01}\right)+\hat{r}_{23.1}\left(\hat{Z}_1-\hat{Z}_{01}\right)}{\hat{r}_{31.1}\left(\hat{X}_1-\hat{X}_{01}\right)+\hat{r}_{32.1}\left(\hat{Y}_1-\hat{Y}_{01}\right)+\hat{r}_{33.1}\left(\hat{Z}_1-\hat{Z}_{01}\right)}$$

$$x^\circ_2 = -c\frac{\hat{r}_{11.2}\left(\hat{X}_2-\hat{X}_{02}\right)+\hat{r}_{12.2}\left(\hat{Y}_2-\hat{Y}_{02}\right)+\hat{r}_{13.2}\left(\hat{Z}_2-\hat{Z}_{02}\right)}{\hat{r}_{31.2}\left(\hat{X}_2-\hat{X}_{02}\right)+\hat{r}_{32.2}\left(\hat{Y}_2-\hat{Y}_{02}\right)+\hat{r}_{33.2}\left(\hat{Z}_2-\hat{Z}_{02}\right)}$$

$$y^\circ_2 = -c\frac{\hat{r}_{21.2}\left(\hat{X}_2-\hat{X}_{02}\right)+\hat{r}_{22.2}\left(\hat{Y}_2-\hat{Y}_{02}\right)+\hat{r}_{23.2}\left(\hat{Z}_2-\hat{Z}_{02}\right)}{\hat{r}_{31.2}\left(\hat{X}_2-\hat{X}_{02}\right)+\hat{r}_{32.2}\left(\hat{Y}_2-\hat{Y}_{02}\right)+\hat{r}_{33.2}\left(\hat{Z}_2-\hat{Z}_{02}\right)}$$

<u>External parameters:</u>

projection center coordinates: X_{01}, Y_{01}, Z_{01} $\quad$ X_{02}, Y_{02}, Z_{02}

sensor angles: ω_1, φ_1, k_1 $\quad$ ω_2, φ_2, k_2

first image parameters: $t_x, t_y, t_z, \Omega, \Phi, K$

second image parameters: $t_x + \Delta x,\ t_y + \Delta y,\ t_z + \Delta z,\ \Omega + \Delta\omega,\ \Phi + \Delta\varphi,\ K + \Delta k$

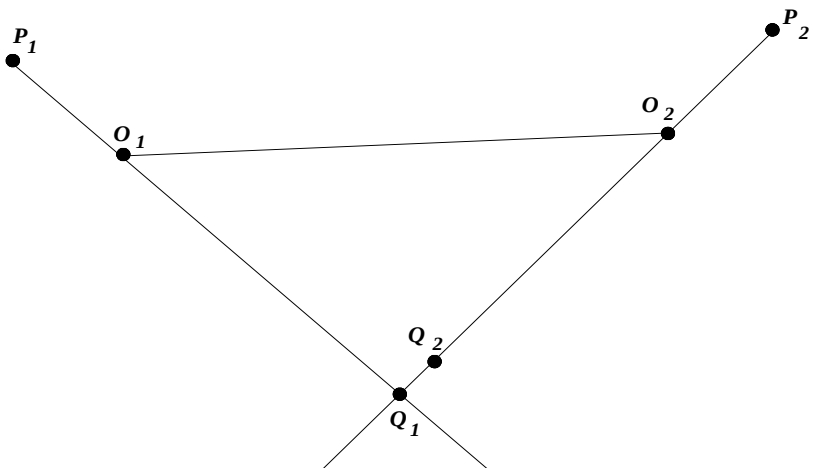
$t_x + \lambda b_x,\ t_y + \lambda b_y,\ t_z + \lambda b_z,\ \Omega + \omega_2,\ \Phi + \varphi_2,\ K + k_2$

<u>Relative orientation parameters:</u> $b_y, b_z, \omega_2, \varphi_2, k_2$ (asymmetric case)

$\varphi_1, k_1, \omega_2, \varphi_2, k_2$ being: $\omega_1 = 0_1$ (symmetric case)

<u>Absolute orientation parameters:</u> $t_x, t_y, t_z, \lambda, \Omega, \Phi, K$

Coplanarity equations – relative orientation



Asymmetric case:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b_x & x_1 & b_x+\xi_2 \\ 0 & b_y & y_1 & b_y+\eta_2 \\ 0 & b_z & -c & b_z+\zeta_2 \end{vmatrix} = 0 \qquad \begin{vmatrix} b_x & x_1 & \xi_2 \\ b_y & y_1 & \eta_2 \\ b_z & -c & \zeta_2 \end{vmatrix} = 0 \qquad \begin{vmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{vmatrix} = R_2^T(\omega_2,\varphi_2,k_2) \begin{vmatrix} x_2 \\ y_2 \\ -c \end{vmatrix}$$

$$\bar{b}_x(y_1\zeta_2 + c\eta_2) - b_y(x_1\zeta_2 + c\xi_2) + b_z(x_1\eta_2 - y_1\xi_2) = 0$$

$$\bar{b}_x[y_1(r_{13}x_2 + r_{23}y_2 - r_{33}c) + c(r_{12}x_2 + r_{22}y_2 - r_{32}c)] +$$
$$-b_y[x_1(r_{13}x_2 + r_{23}y_2 - r_{33}c) + c(r_{11}x_2 + r_{21}y_2 - r_{31}c)] +$$
$$+b_z[x_1(r_{12}x_2 + r_{22}y_2 - r_{32}c) + y_1(r_{11}x_2 + r_{21}y_2 - r_{31}c)] = 0$$

- linearization of this expression by a first order Taylor's expansions:

$$\tilde{C} + \left(\frac{\partial C}{\partial b_y}\right)\delta b_y + \left(\frac{\partial C}{\partial b_z}\right)\delta b_z + \left(\frac{\partial C}{\partial \omega_2}\right)\delta\omega_2 + \left(\frac{\partial C}{\partial \varphi_2}\right)\delta\varphi_2 + \left(\frac{\partial C}{\partial k_2}\right)\delta k_2 = 0$$

$$\frac{\partial C}{\partial b_y} = -\left(x_1\tilde{\zeta}_2 + c\tilde{\xi}_2\right) \qquad \frac{\partial C}{\partial b_z} = x_1\tilde{\eta}_2 + y_1\tilde{\xi}_2$$

$$\frac{\partial C}{\partial \omega_2} = \bar{b}_x[y_1(\tilde{r}_{12}x_2 + \tilde{r}_{22}y_2 - r_{32}c) + c(-\tilde{r}_{13}x_2 - \tilde{r}_{23}y_2 + \tilde{r}_{33}c)] +$$
$$-\tilde{b}_y x_1(\tilde{r}_{12}x_2 + \tilde{r}_{22}y_2 - r_{32}c) + \tilde{b}_z x_1(-\tilde{r}_{13}x_2 - \tilde{r}_{23}y_2 + \tilde{r}_{33}c)$$

$$\frac{\partial C}{\partial \varphi_2} = \left(\tilde{b}_x y_1 - \tilde{b}_y x_1\right)\left(-\cos\tilde{\omega}_2\cos\tilde{\varphi}_2\cos\tilde{k}_2 x_2 + \cos\tilde{\omega}_2\cos\tilde{\varphi}_2 \sin\tilde{k}_2 y_2 + \cos\tilde{\omega}_2\sin\tilde{\varphi}_2 c\right) +$$
$$+\left(\tilde{b}_x c + \tilde{b}_z x_1\right)\left(\sin\tilde{\omega}_2\cos\tilde{\varphi}_2\cos\tilde{k}_2 x_2 - \sin\tilde{\omega}_2\cos\tilde{\varphi}_2\sin\tilde{k}_2 y_2 - \sin\tilde{\omega}_2\sin\tilde{\varphi}_2 c\right) +$$
$$-\left(b_y c + b_z y_1\right)\left(-\sin\tilde{\varphi}_2\cos\tilde{k}_2 x_2 + \sin\varphi_2 \sin k_2 y_2 - \cos\varphi_2 c\right)$$

$$\frac{\partial C}{\partial k_2} = \tilde{b}_x[y_1(\tilde{r}_{23}x_2 - \tilde{r}_{13}y_2) + c(\tilde{r}_{22}x_2 - \tilde{r}_{12}y_2)] - \tilde{b}_y[x_1(\tilde{r}_{23}x_2 - \tilde{r}_{13}y_2) + c(\tilde{r}_{21}x_2 - \tilde{r}_{11}y_2)] +$$
$$+\tilde{b}_z[\tilde{x}_1(\tilde{r}_{22}x_2 - \tilde{r}_{12}y_2) - y_1(\tilde{r}_{21}x_2 - \tilde{r}_{11}y_2)]$$

- **elementary case:** $v = \Delta y = \frac{x_2 - x_1}{\bar{b}_x c}\hat{b}_y + \frac{x_2 - x_1}{\bar{b}_x c}\bar{y}\hat{b}_z - c\left(1 + \frac{\bar{y}^2}{c^2}\right)\hat{\omega}_2 + \frac{x_2\bar{y}}{c}\hat{\varphi}_2 - x_2\hat{k}_2$

with the approximation: $\tilde{R} = \begin{vmatrix} 1 & k & -\varphi \\ -k & 1 & \omega \\ \varphi & -\omega & 1 \end{vmatrix}$

Symmetric case:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & \bar{b}_x & \xi_1 & \bar{b}_x + \xi_2 \\ 0 & 0 & \eta_1 & \eta_2 \\ 0 & 0 & \zeta_1 & \zeta_2 \end{vmatrix} = 0 \qquad \begin{vmatrix} \bar{b}_x & \xi_1 & \xi_2 \\ 0 & \eta_1 & \eta_2 \\ 0 & \zeta_1 & \zeta_2 \end{vmatrix} = 0$$

$$\eta_1 \zeta_2 - \zeta_1 \eta_2 = 0 \qquad \begin{vmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{vmatrix} = R_1^T(\omega_1 = 0, \varphi_1, k_1) \begin{vmatrix} x_1 \\ y_1 \\ -c \end{vmatrix} \qquad \begin{vmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{vmatrix} = R_2^T(\omega_2, \varphi_2, k_2) \begin{vmatrix} x_2 \\ y_2 \\ -c \end{vmatrix}$$

$$\begin{aligned} &(r_{12.1}x_1 + r_{22.1}y_1 - r_{32.1}c)(r_{13.2}x_2 + r_{23.2}y_2 - r_{33.2}c) + \\ &-(r_{13.1}x_1 + r_{23.1}y_1 - r_{33.1}c)(r_{12.2}x_2 + r_{22.2}y_2 - r_{32.2}c) = 0 \end{aligned}$$

- linearization of this expression by a first order Taylor's expansions:

$$\tilde{C} + \left(\frac{\partial C}{\partial \varphi_1}\right)\delta b_y + \left(\frac{\partial C}{\partial k_1}\right)\delta b_z + \left(\frac{\partial C}{\partial \omega_2}\right)\delta \omega_2 + \left(\frac{\partial C}{\partial \varphi_2}\right)\delta \varphi_2 + \left(\frac{\partial C}{\partial k_2}\right)\delta k_2 = 0$$

$$\begin{aligned} \frac{\partial C}{\partial \varphi_1} &= (\tilde{r}_{13.2}x_2 + \tilde{r}_{23.2}y_2 - \tilde{r}_{33.2}c)(-\cos\tilde{\omega}_1 \cos\tilde{\varphi}_1 \cos\tilde{k}_1 x_1 - \sin\tilde{\omega}_1 \cos\tilde{\varphi}_1 \sin\tilde{k}_1 y_1 - \sin\varphi_1 \sin k_1 c) + \\ &\quad -(\tilde{r}_{12.2}x_2 + \tilde{r}_{22.2}y_2 - \tilde{r}_{32.2}c)(-\cos\tilde{\omega}_1 \cos\tilde{\varphi}_1 \cos\tilde{k}_1 x_1 + \cos\tilde{\omega}_1 \cos\tilde{\varphi}_1 \sin\tilde{k}_1 y_1 - \cos\tilde{\omega}_1 \sin\tilde{\varphi}_1 c) \\ \frac{\partial C}{\partial k_1} &= (\tilde{r}_{12.2}x_2 + \tilde{r}_{22.2}y_2 - \tilde{r}_{32.2}c)(\tilde{r}_{22.1}x_1 - \tilde{r}_{12.1}y_1) - (\tilde{r}_{13.2}x_2 + \tilde{r}_{23.2}y_2 - \tilde{r}_{33.2}c)(\tilde{r}_{23.1}x_1 - \tilde{r}_{13.1}y_1) \\ \frac{\partial C}{\partial \omega_2} &= (\tilde{r}_{12.1}x_1 + \tilde{r}_{22.1}y_1 - \tilde{r}_{32.1}c)(\tilde{r}_{12.2}x_2 + \tilde{r}_{22.2}y_2 - \tilde{r}_{32.2}c) + \\ &\quad -(\tilde{r}_{13.1}x_1 + \tilde{r}_{23.1}y_1 - \tilde{r}_{33.1}c)(-\tilde{r}_{13.2}x_2 - \tilde{r}_{23.2}y_2 + \tilde{r}_{33.2}c) \\ \frac{\partial C}{\partial \varphi_2} &= (\tilde{r}_{12.1}x_1 + \tilde{r}_{22.1}y_1 - \tilde{r}_{32.1}c)(-\cos\tilde{\omega}_2 \cos\tilde{\varphi}_2 \cos\tilde{k}_2 x_2 + \cos\tilde{\omega}_2 \cos\tilde{\varphi}_2 \sin\tilde{k}_2 y_2 - \cos\tilde{\omega}_2 \sin\tilde{\varphi}_2 c) + \\ &\quad -(\tilde{r}_{13.1}x_1 + \tilde{r}_{23.1}y_1 - \tilde{r}_{33.1}c)(-\cos\tilde{\omega}_2 \cos\tilde{\varphi}_2 \cos\tilde{k}_2 x_2 - \sin\tilde{\omega}_2 \cos\tilde{\varphi}_2 \sin\tilde{k}_2 y_2 - \sin\varphi_2 \sin k_2 c) \\ \frac{\partial C}{\partial k_2} &= (\tilde{r}_{12.1}x_1 + \tilde{r}_{22.1}y_1 - \tilde{r}_{32.1}c)(\tilde{r}_{33.2}x_2 - \tilde{r}_{13.2}y_2) - (\tilde{r}_{13.1}x_1 + \tilde{r}_{23.1}y_1 - \tilde{r}_{33.1}c)(\tilde{r}_{22.2}x_2 - \tilde{r}_{12.2}y_2) \end{aligned}$$

- transformation between the asymmetric and symmetric parameters of the relative orientation:

$$\begin{aligned} b_x &= \cos\varphi_1 \cos k_1 \\ b_y &= \cos\varphi_1 \sin k_1 \\ b_z &= \sin\varphi_1 \end{aligned} \qquad \begin{aligned} \varphi_1 &= \arcsin b_z \\ k_1 &= \arctan \frac{b_y}{b_x} \end{aligned}$$

$$R_2^T\left(\varpi_2\varphi_2k_2\middle|b_xb_z\right)=R_2^T\left(\varpi_2\varphi_2k_2\right)R_1\left(\varpi_1\varphi_1k_1\right)$$

$$R_2^T\left(\varpi_2\varphi_2k_2\right)=R_2^T\left(\varpi_2\varphi_2k_2\middle|b_xb_z\right)R_1^T\left(\varpi_1\varphi_1k_1\right)$$

- **general case:**
  - exhaustive analysis of preliminary values [12]:

    (note: 12.800 systems are solved and only 4 solutions could be valid, differing in ω_2 and k_2 for π)

	φ_1	k_1	ω_2	φ_2	k_2
□-$\pi/2$	○			○	
□-$\pi/4$	●			●	
0	●	●	●	●	●
$\pi/4$	●	●	●	●	●
$\pi/2$	○	●	●	○	●
$3\pi/4$		●	●		●
π		●	●		●
$5\pi/4$		●	●		●
$3\pi/2$		●	●		●
$7\pi/4$		●	●		●

○ $k_1\equiv 0$ *if* $\varphi_1\equiv\pm\pi/2$ *and/or* $k_2\equiv 0$ *if* $\varphi_2\equiv\pm\pi/2$

  - generalized coplanarity equations [13]:

$$\hat{x}_1-x^\circ{}_1=\hat{u}_1$$

$$\hat{y}_1-y^\circ{}_1=\hat{v}_1$$

$$\hat{x}_2-x^\circ{}_2=\hat{u}_2$$

$$\hat{y}_2-y^\circ{}_2=\hat{v}_2$$

$$\begin{vmatrix}\bar{b}_x & \hat{b}_y & \hat{b}_z\\ \hat{\xi}_1 & \hat{\eta}_1 & \hat{\zeta}_1\\ \hat{\xi}_2 & \hat{\eta}_2 & \hat{\zeta}_2\end{vmatrix}=0 \qquad \begin{vmatrix}\hat{\xi}_1\\ \hat{\eta}_1\\ \hat{\zeta}_1\end{vmatrix}=R_1^T\left(\tilde{\omega}_1,\tilde{\varphi}_1,\tilde{k}_1\right)\begin{vmatrix}\hat{x}_1\\ \hat{y}_1\\ -c\end{vmatrix} \qquad \begin{vmatrix}\hat{\xi}_2\\ \hat{\eta}_2\\ \hat{\zeta}_2\end{vmatrix}=R_2^T\left(\hat{\omega}_2,\hat{\varphi}_2,\hat{k}_2\right)\begin{vmatrix}\hat{x}_2\\ \hat{y}_2\\ -c\end{vmatrix}$$

Model coordinates

$$\begin{vmatrix}x\\ y\\ -c\end{vmatrix}_{ij}=\lambda_{ij}R_j\left(\omega_2,\varphi_2,k_2\right)\left(\begin{vmatrix}X\\ Y\\ Z\end{vmatrix}_i-\begin{vmatrix}X_0\\ Y_0\\ Z_0\end{vmatrix}_j\right) \qquad \frac{1}{\lambda_{ij}}R_j^T\begin{vmatrix}x\\ y\\ -c\end{vmatrix}_{ij}=\begin{vmatrix}X\\ Y\\ Z\end{vmatrix}_i-\begin{vmatrix}X_0\\ Y_0\\ Z_0\end{vmatrix}_j$$

[12] This solutions are performed using the coplanarity equations of the symmetric case.

[13] This solutions are performed using the coplanarity equations of the asymmetric case.

External direction tangents, then referred to the first and second images:

$$\frac{X_i - X_{0j}}{Z_i - Z_{0j}} = \frac{r_{11j}x_{ij} + r_{21j}y_{ij} - r_{31j}c}{r_{13j}x_{ij} + r_{23j}y_{ij} - r_{33j}c}$$

$$\frac{Y_i - Y_{0j}}{Z_i - Z_{0j}} = \frac{r_{12j}x_{ij} + r_{22j}y_{ij} - r_{32j}c}{r_{13j}x_{ij} + r_{23j}y_{ij} - r_{33j}c}$$

- **elementary case:**

$$\frac{X}{Z-h} = \frac{x_1}{-c}; \qquad \frac{Y}{Z-h} = \frac{y_1}{-c}$$

$$\frac{X - \bar{b}_x}{Z - (h + b_z)} = \frac{r_{11}x_2 + r_{21}y_2 - r_{31}c}{r_{13}x_2 + r_{23}y_2 - r_{33}c} \qquad \frac{Y - \bar{b}_y}{Z - (h + b_z)} = \frac{r_{12}x_2 + r_{22}y_2 - r_{32}c}{r_{13}x_2 + r_{23}y_2 - r_{33}c}$$

$$\hat{Z} = h + \frac{\bar{b}_x - b_z \dfrac{\hat{r}_{11}x^\circ{}_2 + \hat{r}_{21}y^\circ{}_2 - \hat{r}_{31}c}{\hat{r}_{13}x^\circ{}_2 + \hat{r}_{23}y^\circ{}_2 - \hat{r}_{33}c}}{\dfrac{x^\circ{}_1}{-c} - \dfrac{\hat{r}_{11}x^\circ{}_2 + \hat{r}_{21}y^\circ{}_2 - \hat{r}_{31}c}{\hat{r}_{13}x^\circ{}_2 + \hat{r}_{23}y^\circ{}_2 - \hat{r}_{33}c}}$$

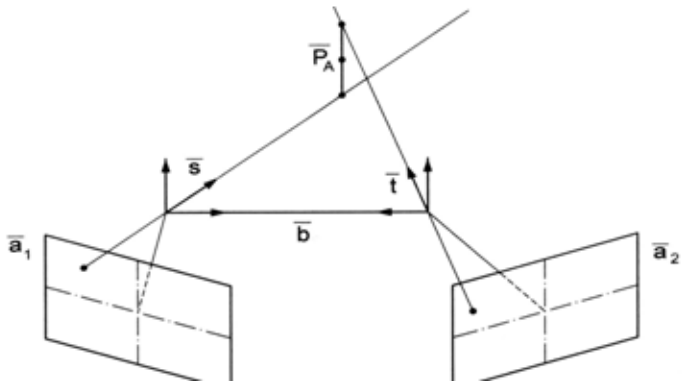
$$\hat{X} = \left(\hat{Z} - h\right)\frac{x^\circ{}_1}{-c}$$

$$\hat{Y}_1 = \left(\hat{Z} - h\right)\frac{y^\circ{}_1}{-c} \qquad \hat{Y}_2 = \hat{b}_y + \left(\hat{Z} - \left(h + \hat{b}_z\right)\right)\frac{\hat{r}_{12}x^\circ{}_2 + \hat{r}_{22}y^\circ{}_2 - \hat{r}_{32}c}{\hat{r}_{13}x^\circ{}_2 + \hat{r}_{23}y^\circ{}_2 - \hat{r}_{33}c}$$

$\hat{V} = \hat{Y}_2 - \hat{Y}_1$ residual parallax

(note: $\hat{X} = u^\circ$ $\hat{Y} = v^\circ$ $\hat{Z} = w^\circ$ in the follows)

- **general case:**



$$\mathbf{A}_1 = \hat{\lambda}\mathbf{s} + \frac{1}{2}\hat{\mathbf{p}} \qquad \mathbf{A}_2 = \hat{\mu}\mathbf{t} - \frac{1}{2}\hat{\mathbf{p}} \qquad \hat{\mathbf{p}} = \hat{\lambda}\mathbf{s} - (\mathbf{b} + \hat{\mu}\mathbf{t})$$

where: $\hat{\mu} = \dfrac{(\mathbf{s}\cdot\mathbf{b})(\mathbf{s}\cdot\mathbf{t})-(\mathbf{b}\cdot\mathbf{t})}{1-(\mathbf{s}\cdot\mathbf{t})^2}$ $\hat{\lambda} = \dfrac{(\mathbf{s}\cdot\mathbf{b})-(\mathbf{b}\cdot\mathbf{t})(\mathbf{s}\cdot\mathbf{t})}{1-(\mathbf{s}\cdot\mathbf{t})^2}$

being: $\mathbf{s} = -\dfrac{\mathbf{a}_1}{|\mathbf{a}_1|}$ $\mathbf{t} = \dfrac{\mathbf{a}_2}{|\mathbf{a}_2|}$

$$\mathbf{a}_1 = \mathbf{R}_1^T \begin{vmatrix} x_1 \\ y_1 \\ -c \end{vmatrix} \qquad \mathbf{a}_2 = \mathbf{R}_2^T \begin{vmatrix} x_2 \\ y_2 \\ -c \end{vmatrix}$$

Absolute orientation

$$\begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} = \hat{\Lambda}_j R_j \left(\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i - \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j \right)$$

- linearization of this expression by a first order Taylor's expansions:

$$\begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} = \begin{vmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{vmatrix}_{ij} + \delta\hat{\lambda}_{ij} R_j \left(\begin{vmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{vmatrix} - \begin{vmatrix} \tilde{t}_x \\ \tilde{t}_y \\ \tilde{t}_z \end{vmatrix} \right) + \tilde{\Lambda}_j \tilde{R}_j \left(\begin{vmatrix} \delta\hat{X} \\ \delta\hat{Y} \\ \delta\hat{Z} \end{vmatrix} - \begin{vmatrix} \delta\hat{t}_x \\ \delta\hat{t}_y \\ \delta\hat{t}_z \end{vmatrix} \right) + \tilde{\Lambda}_j \left(\tilde{R}_\Omega \delta\tilde{\Omega} + \tilde{R}_\Phi \delta\tilde{\Phi} + \tilde{R}_K \delta\tilde{K} \right)_j \left(\begin{vmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{vmatrix}_i - \begin{vmatrix} \tilde{t}_x \\ \tilde{t}_y \\ \tilde{t}_z \end{vmatrix} \right)$$

being: $R_\omega = \begin{vmatrix} 0 & -r_{13} & r_{12} \\ 0 & -r_{23} & r_{22} \\ 0 & -r_{33} & r_{32} \end{vmatrix}$ $R_k = \begin{vmatrix} r_{21} & r_{22} & r_{23} \\ -r_{11} & -r_{12} & -r_{13} \\ 0 & 0 & 0 \end{vmatrix}$

$$R_\varphi = \begin{vmatrix} -\sin\Phi\cos K & \sin\Omega\cos\Phi\cos K & -\cos\Omega\cos\Phi\cos K \\ \sin\Phi\sin K & -\sin\Omega\cos\Phi\sin K & \cos\Omega\cos\Phi\sin K \\ \cos\Phi & \sin\Omega\sin\Phi & -\cos\Phi\sin\Phi \end{vmatrix}$$

<u>Three images procedure:</u>

- a bridging between 2 models, starting from $4\times4=16$ possible cases (where 4 are the solution obtained from the general case of the relative orientation), leads to 2 valid solutions (where the first solution is specular respect the second one and vice-versa):

Models	2A	2B	2C	2D
1A	1A – 2A	1A – 2B	1A – 2C	1A –2 D
1B	1B – 2A	1B – 2B	1B – 2C	1B – 2D
1C	1C – 2A	1C – 2B	1C – 2C	1C – 2D
1D	1D – 2A	1D – 2B	1D – 2C	1D – 2D

- a successive absolute orientation individuates the correct solution between the last 2 valid ones.

Quasi-exact solution in the absolute orientation:

$$\begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} = \hat{\lambda}_j \hat{R}_j \left(\hat{\Omega},\hat{\phi},\hat{K}\right) \left(\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i - \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j \right)$$

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \frac{1}{\hat{\lambda}_j} \hat{R}_j^T \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j = \hat{\mu}_j \hat{R}_j^T \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j$$

$$\begin{vmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{vmatrix} = \hat{\mu}_j \hat{R}_j^T \begin{vmatrix} \overline{u}^\circ \\ \overline{v}^\circ \\ \overline{w}^\circ \end{vmatrix}_{ij} + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j$$

$$\begin{vmatrix} \hat{X} - \overline{X} \\ \hat{Y} - \overline{Y} \\ \hat{Z} - \overline{Z} \end{vmatrix}_i = \hat{\mu}_j \hat{R}_j^T \begin{vmatrix} u^\circ - \overline{u}^\circ \\ v^\circ - \overline{v}^\circ \\ w^\circ - \overline{w}^\circ \end{vmatrix}_{ij}$$

$$\hat{\mu}_j^2 = \sqrt{\left\| \left(\hat{X} - \overline{X}\right)^T \quad \left(\hat{Y} - \overline{Y}\right)^T \quad \left(\hat{Z} - \overline{Z}\right)^T \right\| \begin{vmatrix} \hat{X} - \overline{X} \\ \hat{Y} - \overline{Y} \\ \hat{Z} - \overline{Z} \end{vmatrix}_i \Bigg/ \left\| \left(u^\circ - \overline{u}^\circ\right)^T \quad \left(v^\circ - \overline{v}^\circ\right)^T \quad \left(w^\circ - \overline{w}^\circ\right)^T \right\| \begin{vmatrix} u^\circ - \overline{u}^\circ \\ v^\circ - \overline{v}^\circ \\ w^\circ - \overline{w}^\circ \end{vmatrix}_{ij}}$$

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \hat{R}_j^T \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij}$$

being: $\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \begin{vmatrix} \hat{X} - \overline{X} \\ \hat{Y} - \overline{Y} \\ \hat{Z} - \overline{Z} \end{vmatrix}_i \qquad \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} = \hat{\mu}_j \begin{vmatrix} u^\circ - \overline{u}^\circ \\ v^\circ - \overline{v}^\circ \\ w^\circ - \overline{w}^\circ \end{vmatrix}_{ij}$

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \left(I - S_j\right)^{-1} \left(I + S_j\right) \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij}$$

where: $S = \begin{vmatrix} 0 & c_j & -b_j \\ -c_j & 0 & a_j \\ b_j & -a_j & 0 \end{vmatrix}$

$$\left(I - S_j\right) \begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \left(I + S_j\right) \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij}$$

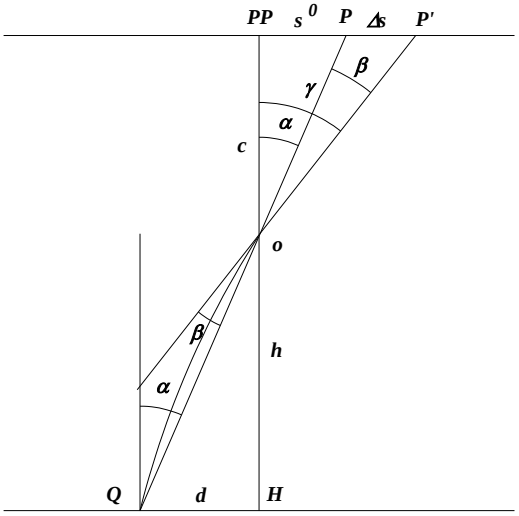
$$\begin{vmatrix} 1 & -c_j & b_j \\ c_j & 1 & -a_j \\ -b_j & a_j & 1 \end{vmatrix} \begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \begin{vmatrix} 1 & c_j & -b_j \\ -c_j & 1 & a_j \\ b_j & -a_j & 1 \end{vmatrix} \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij}$$

$$\begin{vmatrix} 0 & \hat{Z}_i - w^\circ{}_{ij} & -\left(\hat{Y}_i - v^\circ{}_{ij}\right) \\ -\left(\hat{Z}_i - w^\circ{}_{ij}\right) & 0 & \hat{X}_i - u^\circ{}_{ij} \\ \hat{Y}_i - v^\circ{}_{ij} & -\left(\hat{X}_i - u^\circ{}_{ij}\right) & 0 \end{vmatrix} \begin{vmatrix} \hat{a}_j \\ \hat{b}_j \\ \hat{c}_j \end{vmatrix} + \begin{vmatrix} \hat{X}_i - u^\circ{}_{ij} \\ \hat{Y}_i - v^\circ{}_{ij} \\ \hat{Z}_i - w^\circ{}_{ij} \end{vmatrix} = 0$$

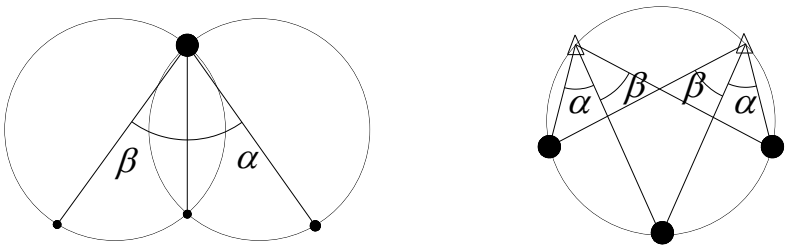
$$\begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j = \begin{vmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{vmatrix} - \hat{\mu}_j \hat{R}_j^T \begin{vmatrix} \overline{u}^\circ \\ \overline{v}^\circ \\ \overline{w}^\circ \end{vmatrix}_{ij_i} \qquad R^T = \frac{1}{1 + a^2 + b^2 + c^2} \begin{vmatrix} 1 + a^2 - b^2 - c^2 & 2(ab + c) & 2(ac - b) \\ 2(ab - c) & 1 - a^2 + b^2 - c^2 & 2(bc + a) \\ 2(ac + b) & 2(bc - a) & 1 - a^2 - b^2 + c^2 \end{vmatrix}$$

Atmospheric refraction and earth curvature effects

$$\Delta s = \frac{1-K}{2R} hs\left(1+\frac{s^2}{c^2}\right)$$



Critic cylinder in the relative orientation

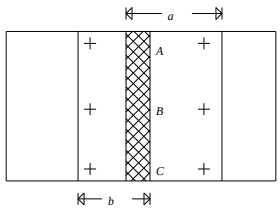


$$\hat{b}_y + \frac{Y}{h}\hat{b}_z - 2R\hat{\omega}_2 + \frac{(b-X)Y}{h}\hat{\varphi}_2 - (b-X)\hat{k}_2 - \Delta Y° = 0$$

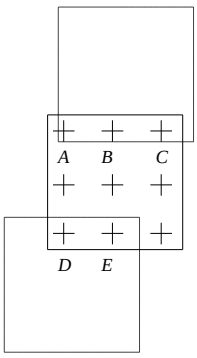
Aerial triangulation

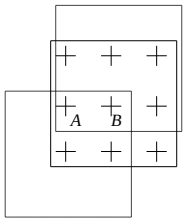
Block adjustment among several images or models, using many tie points and few control points:

- longitudinal overlapping (min 60%):



- transversal overlapping (min 20%) (60%)





Anblock method:

$$\begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} = \hat{\lambda}_j \hat{R}_j\left(\hat{\Omega},\hat{\phi},\hat{K}\right)\left(\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i - \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j\right) \qquad \begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \frac{1}{\hat{\lambda}_j}\hat{R}_j^T \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j$$

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix} = \begin{vmatrix} \hat{a} & -\hat{b} & \frac{\hat{\varphi}}{\hat{\lambda}} \\ \hat{b} & \hat{a} & -\frac{\hat{\Omega}}{\hat{\lambda}} \\ \frac{-\hat{\varphi}\cos\hat{K} + \hat{\Omega}\sin\hat{K}}{\hat{\lambda}} & \frac{\hat{\varphi}\sin\hat{K} + \hat{\Omega}\cos\hat{K}}{\hat{\lambda}} & \frac{1}{\hat{\lambda}} \end{vmatrix}_j \begin{vmatrix} u^\circ \\ v^\circ \\ w^\circ \end{vmatrix}_{ij} + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j$$

where: $\hat{a}_j = \cos\hat{K}_j / \hat{\lambda}_j \qquad \frac{1}{\hat{\lambda}_j} = \sqrt{\hat{a}_j^2 + \hat{b}_j^2}$

$$\hat{b}_j = \sin\hat{K}_j / \hat{\lambda}_j \qquad \hat{K}_j = \arctan\frac{\hat{b}_j}{\hat{a}_i}$$

$$\hat{R}_j^T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\hat{\Omega} \\ 0 & \hat{\Omega} & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & \hat{\phi} \\ 0 & 1 & 0 \\ -\hat{\phi} & 0 & 1 \end{vmatrix} \begin{vmatrix} \cos\hat{K} & -\sin\hat{K} & 0 \\ \sin\hat{K} & \cos\hat{K} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\hat{K} & -\sin\hat{K} & \hat{\phi} \\ \sin\hat{K} & \cos\hat{K} & -\hat{\Omega} \\ -\hat{\phi}\cos\hat{K} + \hat{\Omega}\sin\hat{K} & \hat{\phi}\sin\hat{K} + \hat{\Omega}\cos\hat{K} & 1 \end{vmatrix}$$

- splitting in two groups the horizontal and vertical parameters

$\hat{X}_i, \hat{Y}_i \qquad \hat{t}_x, \hat{t}_y \qquad \hat{\lambda}_j, \hat{K}_j$ (i.e.: $\hat{a}_j, \hat{b}_j$)

$\hat{Z}_i \qquad \hat{t}_{z_j} \qquad \hat{\Omega}_j, \hat{\Phi}_j$

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{vmatrix}_i = \begin{vmatrix} u^\circ_{ij} & -v^\circ_{ij} & 0 & w^\circ_{ij}/\hat{\lambda}_j \\ v^\circ_{ij} & u^\circ_{ij} & -w^\circ_{ij}/\hat{\lambda}_j & 0 \\ 0 & 0 & \frac{u^\circ_{ij}\sin\hat{K}_j + v^\circ_{ij}\sin\hat{K}_j}{\hat{\lambda}_j} & \frac{-u^\circ_{ij}\sin\hat{K}_j + v^\circ_{ij}\sin\hat{K}_j}{\hat{\lambda}_j} \end{vmatrix} \begin{vmatrix} \hat{a} \\ \hat{b} \\ \hat{\Omega} \\ \hat{\Phi} \end{vmatrix}_j + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \\ \hat{t}_z \end{vmatrix}_j + \begin{vmatrix} 0 \\ 0 \\ w^\circ_{ij}/\hat{\lambda}_j \end{vmatrix}$$

- horizontal equations:

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \end{vmatrix}_i = \begin{vmatrix} u^\circ & -v^\circ \\ v^\circ & u^\circ \end{vmatrix}_{ij} \begin{vmatrix} \hat{a} \\ \hat{b} \end{vmatrix}_j + \begin{vmatrix} \hat{t}_x \\ \hat{t}_y \end{vmatrix}_j + w^\circ_{ij} \begin{vmatrix} \tilde{\phi}/\tilde{\lambda} \\ -\tilde{\Omega}/\tilde{\lambda} \end{vmatrix}_j$$

where: $\tilde{\Omega} = \tilde{\Phi} = 0$ (at the first step)

$$\begin{vmatrix} \hat{X} \\ \hat{Y} \end{vmatrix}_i = \begin{vmatrix} X^\circ \\ Y^\circ \end{vmatrix}_i$$

(for the control points)

- vertical equations:

$$\left|\hat{Z}\right|_i = \left\| \begin{bmatrix} \sin \tilde{K}/\tilde{\lambda} & \cos \tilde{K}/\tilde{\lambda} \\ -\cos \tilde{K}/\tilde{\lambda} & \sin \tilde{K}/\tilde{\lambda} \end{bmatrix}_j \begin{bmatrix} u^\circ \\ v^\circ \end{bmatrix} \right\|_{ij}^T \left|\begin{matrix} \hat{\Omega} \\ \hat{\Phi} \end{matrix}\right|_j + \left|\hat{t}_z\right|_j + w^\circ{}_{ij} \left|1/\hat{\lambda}_j\right|$$

$$\left|\hat{Z}\right|_i = \left|Z^\circ\right|_i$$ (for the control points)

Image analysis

- grey intensity: $g_i = \dfrac{i-1}{N-1} \qquad i = 1,\ldots,N$
- red intensity: $r_i = \dfrac{i-1}{M-1} \qquad i = 1\ldots M$
- green intensity: $g_j = \dfrac{j-1}{M-1} \qquad j = 1\ldots M$
- blue intensity: $b_k = \dfrac{k-1}{M-1} \qquad i = 1\ldots M$

<u>Re-sampling:</u>

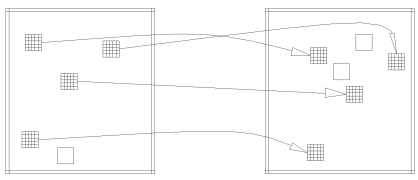
$$g^2(i,j) = s^2 \sum_{k=1}^{N} \sum_{l=1}^{N} K(i',j',k,l) g(k,l)$$ where: $K(i',j',k,l) = \begin{cases} 0 & if \quad |i'-k| < \frac{2}{s} \cap |j'-l| < \frac{2}{s} \\ 1 & otherwise \end{cases}$

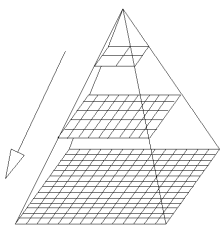
$$g^s(i,j) = \sum_{k=1}^{N} \sum_{l=1}^{N} K(i',j',k,l) g(k,l)$$ where: $K(i',j',k,l) = \dfrac{\sin(i'-k)}{i'-k} \cdot \dfrac{\sin(j'-l)}{j'-l}$

(note: Nyquist frequency: $f_{\max} = \dfrac{1}{2\Delta}\, cycles/pixel$)

<u>Image matching:</u>

- homology correspondence and image pyramid:





- least square adjustment [14]: $\sum_i \sum_j \left(g_p(i,j) - g_r(i,j)\right)^2 = min$
- affine transformation: $g_p(i,j) = c_0 + c_1 g_r(i + \Delta r + ic_2, j + \Delta c + ic_4 + jc_5)$

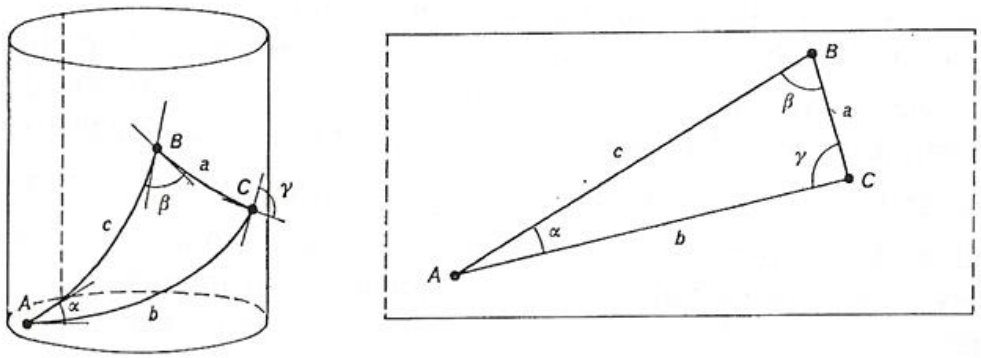
[14] Robust procedures, as well as different transformation (considering the following point) are possible alternatives.

PART III – CARTOGRAPHY

From a spherical or ellipsoid object to a plane representation

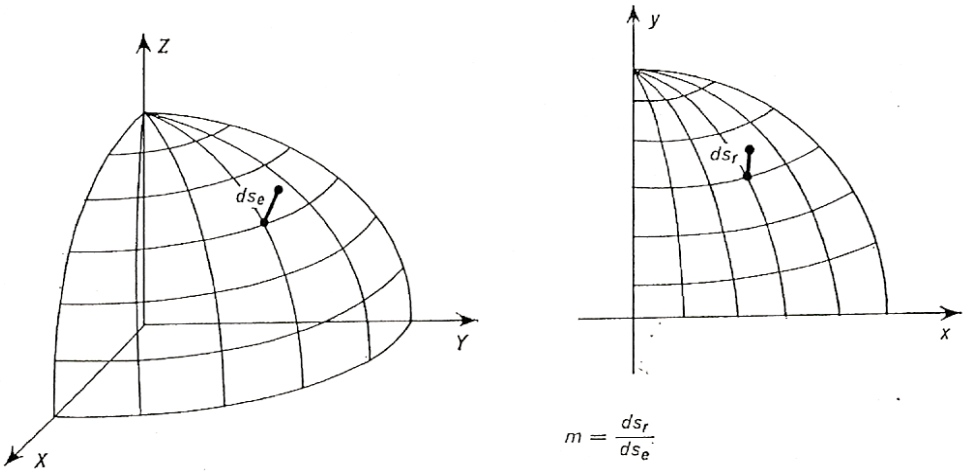
- Map projection equations: $x = x(\varphi,\lambda)$ $\quad$ $x = x(\varphi,\lambda)$

  $y = y(\varphi,\lambda)$ $\quad$ $y = y(\varphi,\lambda)$

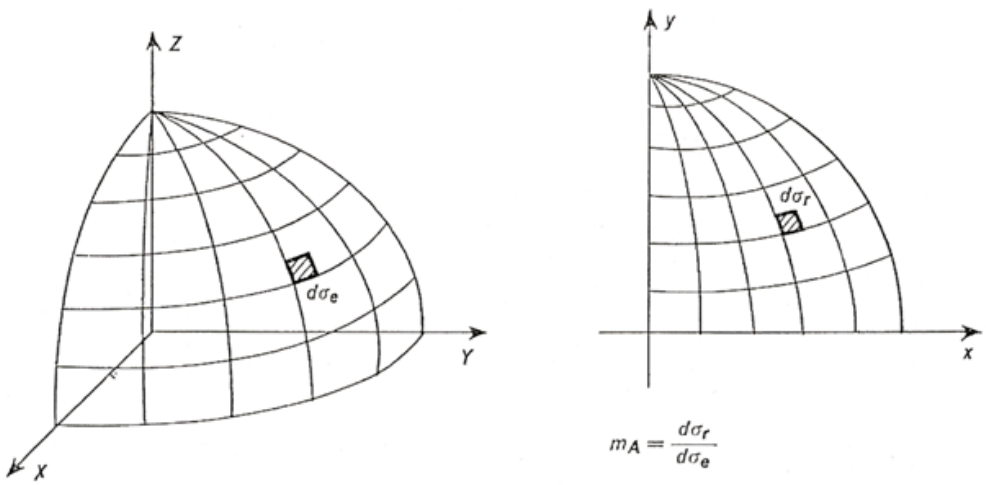


Development of triangle from a one-curvature surface to a plane

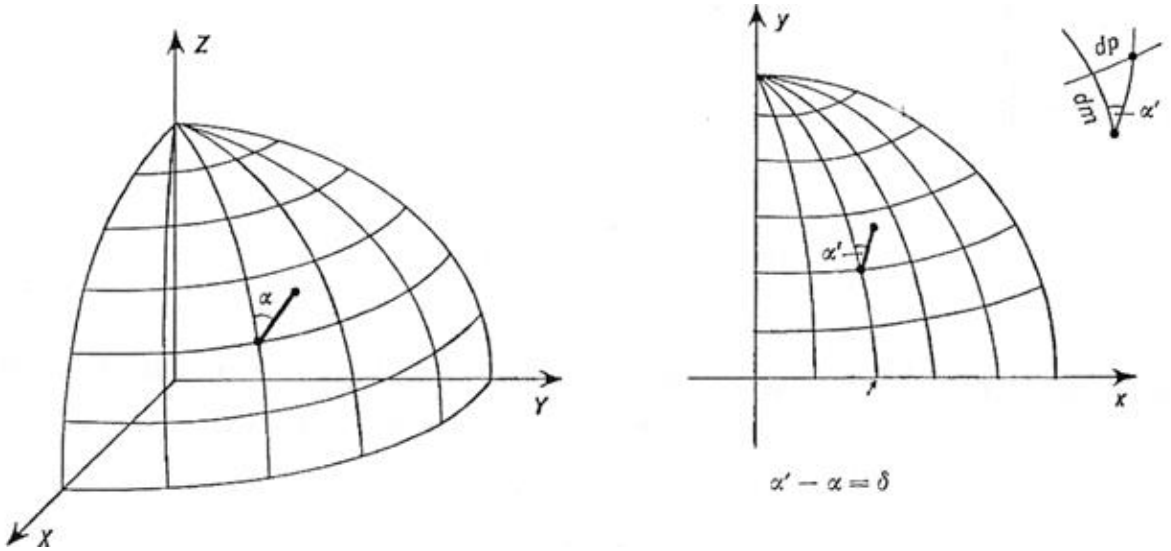
<u>Types of deformation starting from a two-curvature surface</u>



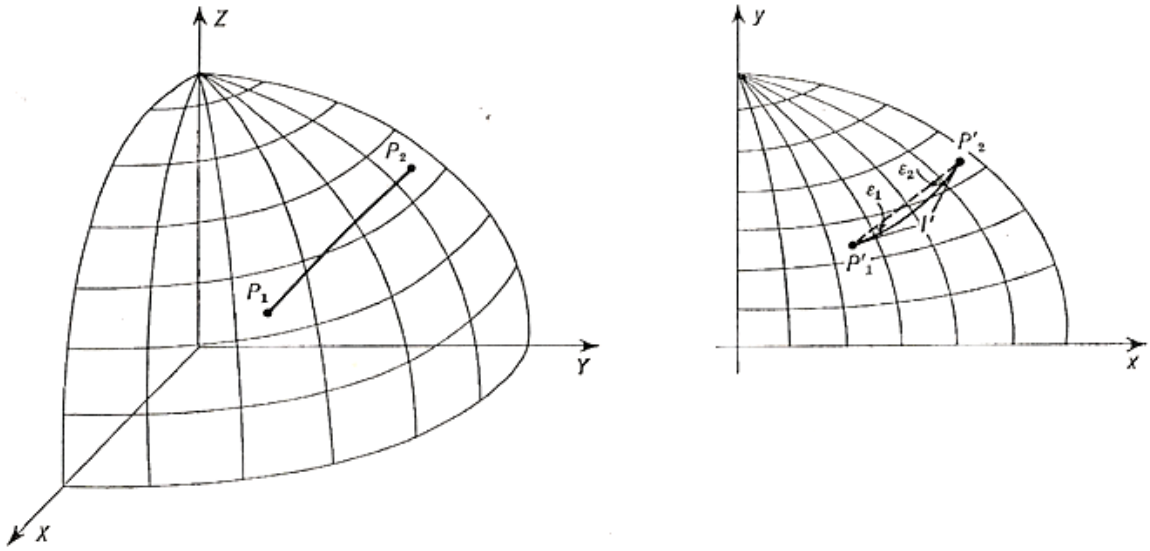
Linear deformation



Areal deformation

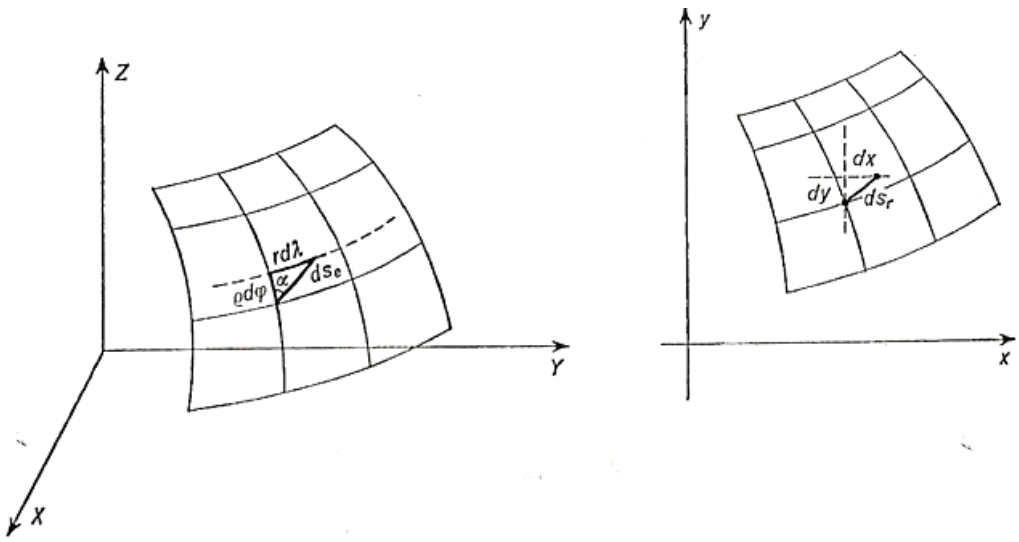


Angular deformation and meridian transformation



Geodetic line and its representation

<u>Metric evaluation of the cartographic equivalencies</u>



$$ds_e^2 = r^2 d\lambda^2 + \rho^2 d\varphi^2$$

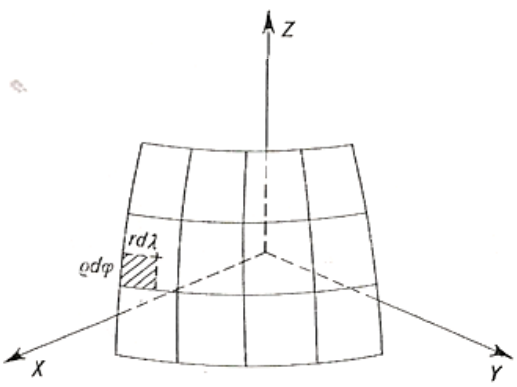
$$ds_r^2 = dx^2 + dy^2 = e^2 d\varphi^2 + 2f\, d\varphi\, d\lambda + g^2 d\lambda^2$$

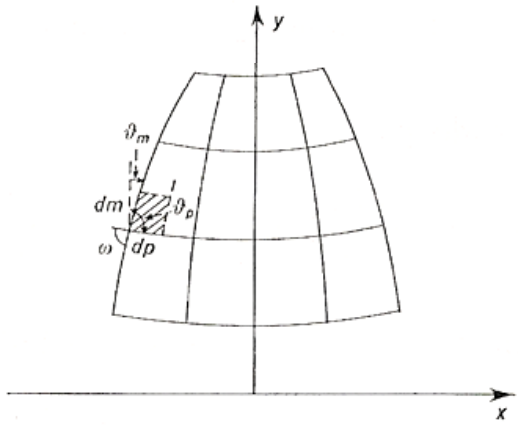
where: $dx = \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial \lambda} d\lambda \qquad dy = \frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial \lambda} d\lambda$

$$e = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 \qquad f = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} \qquad g = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2.$$

$$e/\rho^2 = e^* \qquad f/r\rho = f^* \qquad g/r^2 = g^*$$

$$m^2 = \frac{e}{\rho^2}\cos^2\alpha + \frac{2f}{r\rho}\sin\alpha\cos\alpha + \frac{g}{r^2}\sin^2\alpha = e^*\cos^2\alpha + 2f^*\sin\alpha\cos\alpha + g^*\sin^2\alpha$$





$$d\sigma_e = \rho\, d\varphi\, r\, d\lambda = r\rho\, d\varphi\, d\lambda$$

$$d\sigma_r = dm\, dp \sin\omega = \rho\, d\varphi \sqrt{e^*}\, r\, d\lambda \sqrt{g^*} \sin\omega$$

where: $dm = \rho\, d\varphi \sqrt{e^*}$

$$m_A = \frac{d\sigma_r}{d\sigma_e} = \frac{\sqrt{eg}}{r\rho}\sin\omega = \sqrt{e^* g^*}\sin\omega$$

moreover: $\tan\theta_m = \frac{dx}{dy} = \frac{\partial x}{\partial \varphi} d\varphi \Big/ \frac{\partial y}{\partial \varphi} d\varphi$

$$\tan\theta_p = \frac{dx}{dy} = \frac{\partial x}{\partial \lambda} d\lambda \Big/ \frac{\partial y}{\partial \lambda} d\lambda$$

$$\tan\omega = \tan(\theta_p - \theta_m) = \frac{\tan\theta_p - \tan\theta_m}{1 + \tan\theta_p \tan\theta_m}$$

$$\cos\omega = \sqrt{1 - sen^2\omega} = f/\sqrt{eg}$$

$$sen\,\omega = \frac{tan\,\omega}{\sqrt{1+tan^2\,\omega}}\frac{1}{\sqrt{eg}}\left(\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda}-\frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda}\right)$$

$$m_A = \frac{1}{r\rho}\left(\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda}-\frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda}\right)$$

finally: $$tan\,\alpha' = \frac{dp}{dm} = \frac{r\,d\lambda\sqrt{g^*}}{\rho\,d\varphi\sqrt{e^*}} = \sqrt{\frac{g}{e}}\frac{d\lambda}{d\varphi} = \frac{\rho}{r}\sqrt{\frac{g}{e}}\,tan\,\alpha$$

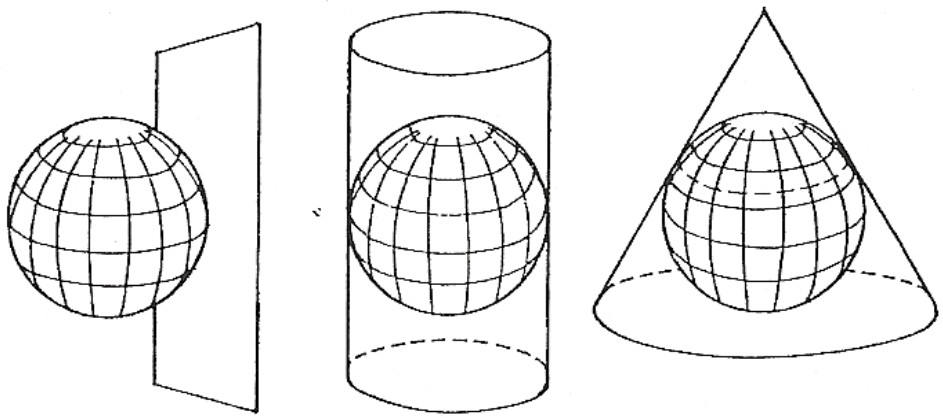
being: $$\rho\,d\varphi = ds_e\,cos\,\alpha$$

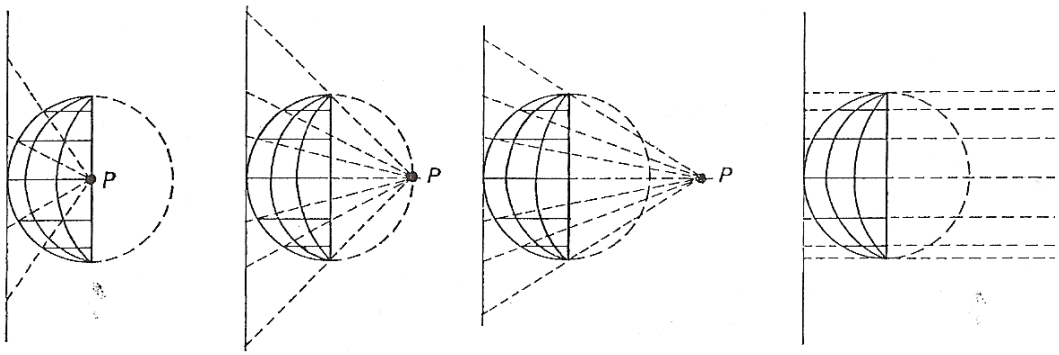
$$r\,d\lambda = ds_e sin\,\alpha$$

$$\frac{d\lambda}{d\varphi} = \frac{\rho}{r}tan\,\alpha$$

$$tan\,\delta = tan(\alpha' - \alpha) = \frac{tan\,\alpha' - tan\,\alpha}{1+tan\,\alpha'\,tan\,\alpha} = \frac{\left(\frac{\rho}{r}\sqrt{\frac{g}{e}}-1\right)tan\,\alpha}{1+\frac{\rho}{r}\sqrt{\frac{g}{e}}\,tan^2\,\alpha}$$

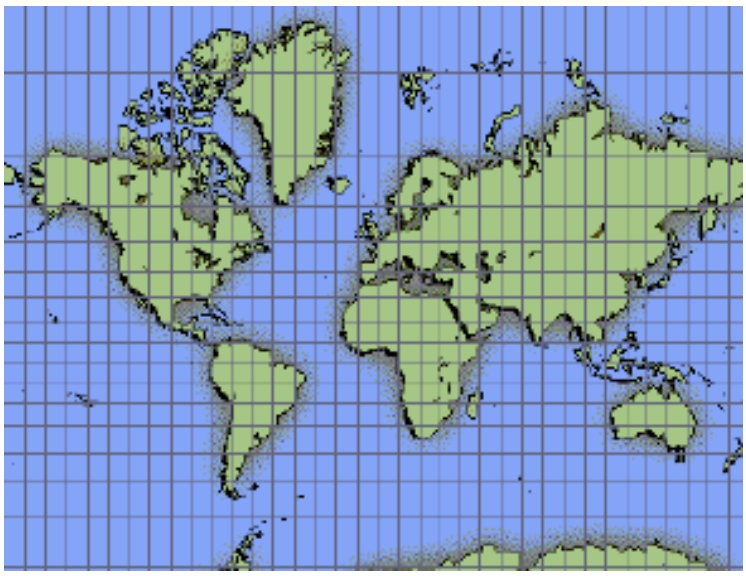
<u>Map projection classification</u>





Map projection examples

Mercator conformal projection



- Map projection equations: $$y = a\,ln\left(\left(\frac{1-e\,sin\varphi}{1+e\,sin\varphi}\right)^{\frac{e}{2}} tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right)$$
$$x = a\lambda$$

- Linear deformation: $$m = \frac{a}{r} = \frac{\left(1-e^2 sin^2\varphi\right)^{\frac{1}{2}}}{\cos\varphi}$$

  being: $$r = \frac{a\,cos\varphi}{\left(1-e^2 sin^2\varphi\right)^{\frac{1}{2}}}$$

- Angular deformation: $$\alpha' - \alpha = 0$$

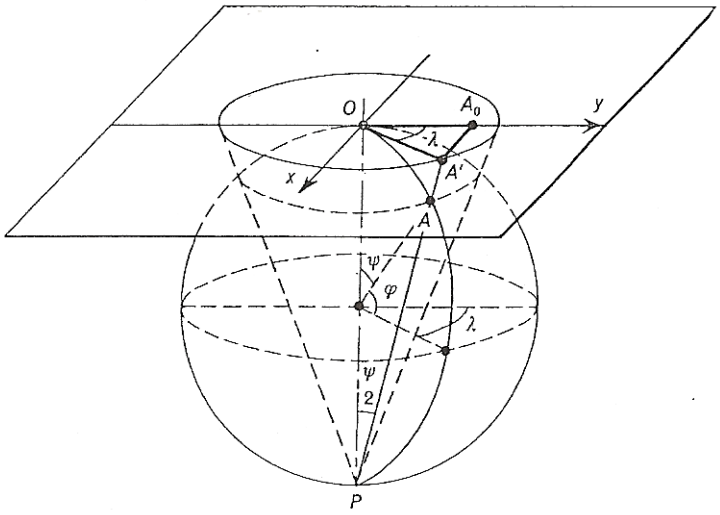
Sanson-Flamsted or natural equivalent projection

- Map projection equations: $$y = \int_{\varphi_0}^{\varphi} \rho\, d\varphi$$
$$x = r(\lambda-\lambda_0) = N\,cos\,\varphi\,(\lambda-\lambda_0)$$

- Linear deformation: $$m^2 = 1-(\lambda-\lambda_0)sin\,\varphi\,sin\,2\alpha + (\lambda-\lambda_0)^2\,sin^2\,\varphi\,cos^2\,\alpha =$$
$$= 1-\frac{1}{2}(\lambda-\lambda_0)sin\,\varphi\,sin\,2\alpha$$

- Areal deformation: $$m_A = 1$$

Universal stereographic polar (UPS) conformal projection

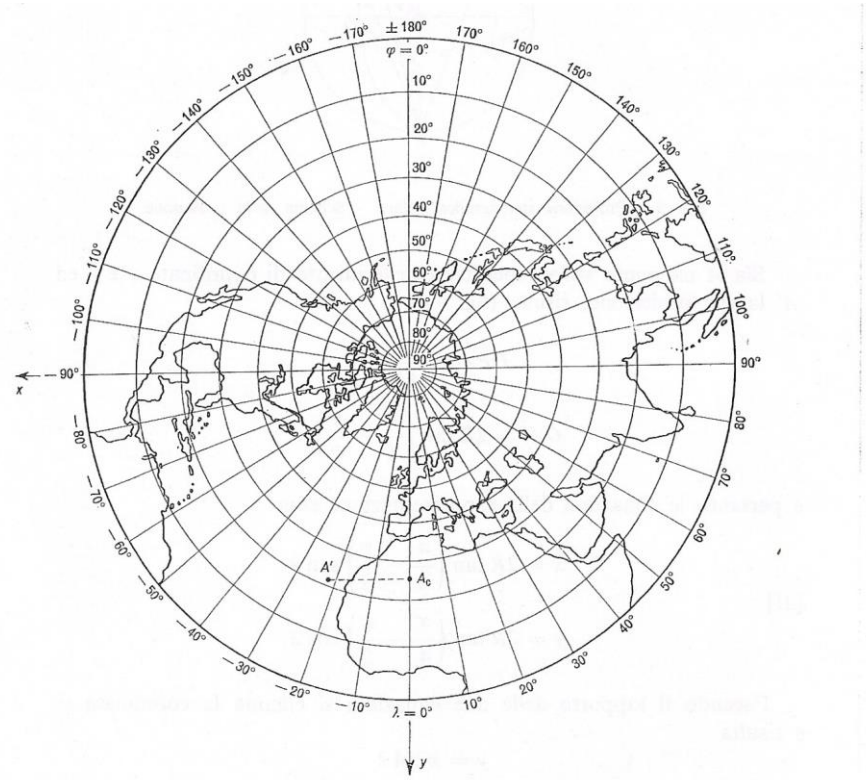


- Map projection equations: $x = 2R\,tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) sin\,\lambda$

$$y = 2R\,tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) cos\,\lambda$$

where: $O\widehat{P}A = \frac{\pi}{4} - \frac{\varphi}{2}$

$$OA' = 2R\,tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$$

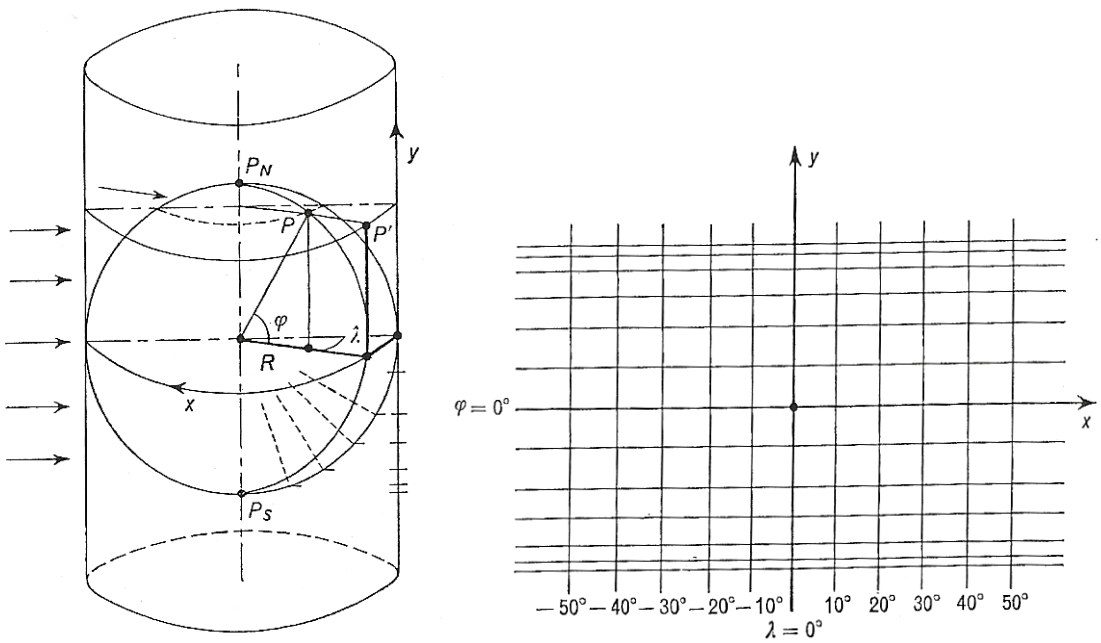


- Linear deformation: $$m = \frac{ds_r}{ds_e} = \frac{1}{cos^2\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)}$$

being: $ds_e = Rd\varphi$, and ds_r:

$$ds_r = \frac{d(OA')}{d\varphi} d\varphi = -\frac{R}{cos^2\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)} d\varphi$$

<u>Lambert cylindrical equivalent projection</u>



- Map projection equations: $x = R\lambda$

$y = R\,sin\,\varphi$

- Areal deformation: $$m_A = \frac{1}{R \cdot R\,cos\,\varphi} R \cdot R\,cos\,\varphi = 1$$

<u>Ptolemy conic quasi-equivalent projection</u>

- Map projection equations: $x = p\,sin\,\vartheta = (a - R\varphi)sin\,b\lambda$

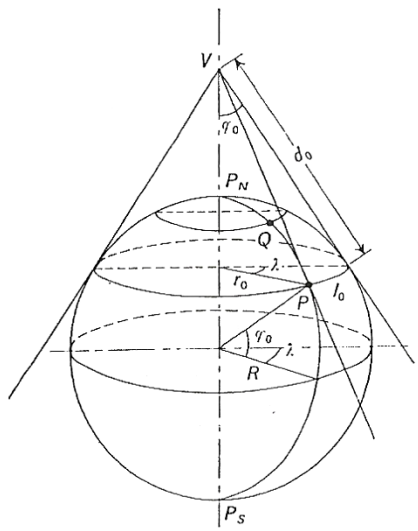
$y = p\,cos\,\vartheta = (a - R\varphi)cos\,b\lambda$

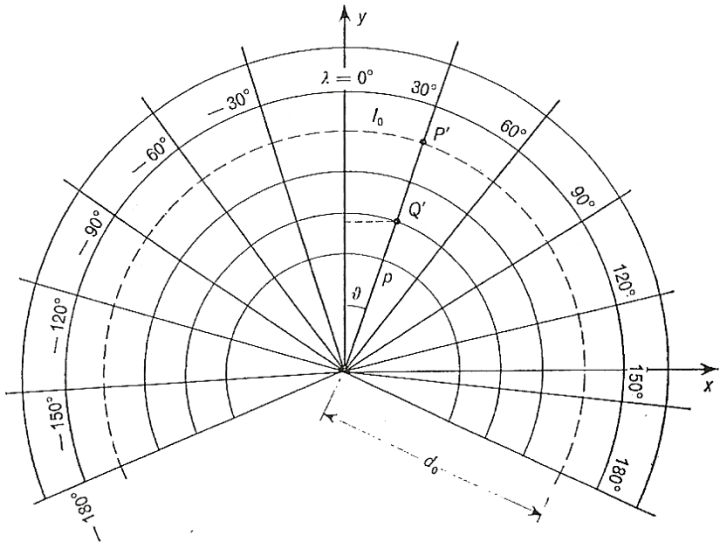
where: $$\vartheta = \frac{R\,cos\,\varphi_0 \cdot \lambda}{R/tan\,\varphi_0} = \lambda\,sin\,\varphi_0 = b\lambda < 2\pi$$

$$p = \frac{R}{tan\,\varphi_0} - R(\varphi - \varphi_0) = a - R\varphi$$

$$a = R\left(\frac{1}{tan\,\varphi_0} + \varphi_0\right)$$

$$b = sin\,\varphi_0$$





<u>Cassini-Soldner polyhedral quasi-equivalent projection</u>

- Map projection equations: $y = Y$ being: $Y = \varphi_m(\varphi_Q - \varphi_0)$

  $x = X$ $X = (\lambda_P - \lambda_0)N_P\,cos\,\varphi_P$

  where: $\varphi_m = \dfrac{\varphi_0 + \varphi_Q}{2}$

  $$\varphi_Q = \varphi_P + \frac{(\lambda_P - \lambda_0)^2 N_P\,sin\,2\varphi_P}{4\rho_P}$$

- Linear deformation: $m = 1 + \dfrac{x^2\,cos^2\,\alpha}{2\rho_0 N_0}$

  $m_{12} = 1 + \dfrac{x_1^2 + x_1x_2 + x_2^2}{6\rho_0 N_0}\,cos^2\,\alpha'_{12}$ (in the segment $1,2$)

- Areal deformation: $m_A = 1 + \dfrac{x^2}{2\rho_0 N_0}$

- Angular deformation: $\alpha'_{12} - \alpha_{12} = -\dfrac{\sigma\,cos\,\alpha'_{12}}{2\rho_0 N_0}\left(x_1 + \dfrac{1}{3}\sigma\,sin\,\alpha'_{12}\right) +$

  $$-\frac{x_1^2 + x_1x_2 + x_2^2}{3} \cdot \frac{sin\,\alpha'_{12}\,cos\,\alpha'_{12}}{2\rho_0 N_0}$$

Differential equations

<u>Conformal map projections</u>

- 1st geometric condition: $\delta = \dfrac{\rho\sqrt{g/e}}{r} - 1 = 0 \quad \Rightarrow \quad g = \dfrac{r^2 e}{\rho^2}$

$$\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = \frac{r^2}{\rho^2}\left(\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2\right)$$

moreover: $u = ln\left(\left(\dfrac{1 - e \cdot sen\varphi}{1 + e \cdot sen\varphi}\right)^{\frac{e}{2}} \sqrt{\dfrac{1 + sen\varphi}{1 - sen\varphi}}\right) = ln\left(\left(\dfrac{1 - e \cdot sen\varphi}{1 + e \cdot sen\varphi}\right)^{\frac{e}{2}} tan\left(\dfrac{\pi}{4} + \dfrac{\varphi}{2}\right)\right)$

being: $du = \dfrac{\rho}{r} d\varphi$

$$u = \int_0^{\varphi} \frac{\rho}{r} d\varphi$$

$$ds_e^2 = r^2\left(du^2 + d\lambda^2\right)$$

$$\Rightarrow \quad \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2$$

where: $\dfrac{\partial x}{\partial \varphi} = \dfrac{\partial x}{\partial u}\dfrac{du}{\partial \varphi} = \dfrac{\rho}{r}\dfrac{\partial x}{\partial u} \qquad \dfrac{\partial y}{\partial \varphi} = \dfrac{\partial y}{\partial u}\dfrac{\partial u}{\partial \varphi} = \dfrac{\rho}{r}\dfrac{\partial y}{\partial u}$

- 2nd geometric condition: $cos\,\omega = \dfrac{f}{\sqrt{eg}} = 0 \quad \Rightarrow \quad f = 0$

$$\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} = 0$$

$$\Rightarrow \quad \frac{\partial x}{\partial u}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial \lambda} = 0$$

- Partial differential equations: $\dfrac{\partial y}{\partial u} = \dfrac{\partial x}{\partial \lambda}$

$$\frac{\partial y}{\partial \lambda} = -\frac{\partial x}{\partial u}$$

- Linear deformation: $m = \dfrac{\sqrt{e}}{\rho}$

- Complex representation: $y + ix = f(u + i\lambda)$

$$y + ix = f(u) + f^{I}(u)i\lambda + \frac{1}{2} f^{II}(u)(i\lambda)^2 + \frac{1}{3!} f^{III}(u)(i\lambda)^3 + \frac{1}{4!} f^{IV}(u)(i\lambda)^4 + \frac{1}{5!} f^{V}(u)(i\lambda)^5 + \ldots =$$

$$= f(u) + f^{I}(u)i\lambda - \frac{1}{2} f^{II}(u)\lambda^2 - \frac{1}{3!} f^{III}(u)i\lambda^3 + \frac{1}{4!} f^{IV}(u)\lambda^4 + \frac{1}{5!} f^{V}(u)i\lambda^5 - \ldots$$

being: i the imaginary unit, so that: $i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$ *etc.*

$$y = f(u) - \frac{1}{2} f^{II}(u)\lambda^2 + \frac{1}{24} f^{IV}(u)\lambda^4 - \ldots$$

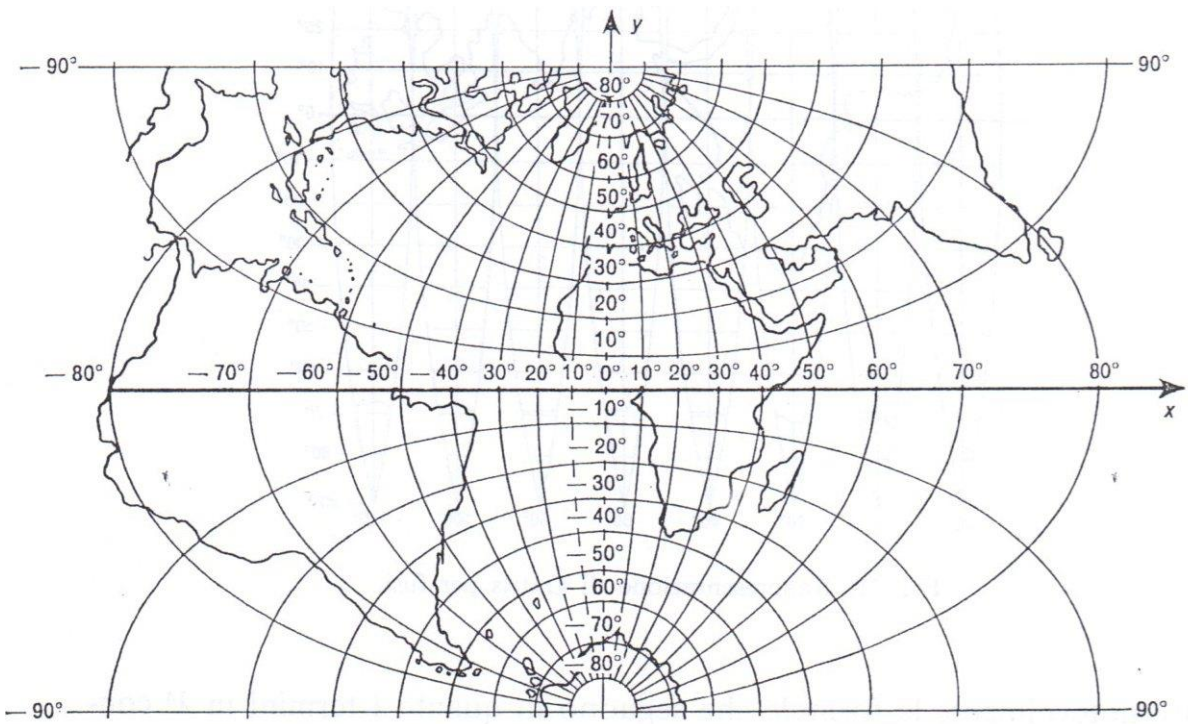
$$x = f^{I}(u)\lambda - \frac{1}{6} f^{III}(u)\lambda^3 + \frac{1}{120} f^{V}(u)\lambda^5 - \ldots$$

<u>Equivalent map projection</u>

- Geometric condition: $m_A = 1$
- Partial differential equations: $\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda} = r\rho$

$$\Rightarrow \quad \frac{\partial y}{\partial u}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial \lambda} = r^2$$

Gauss universal transversal (cylindrical) Mercator (UTM) conformal projection



- Geometric condition: $y_{(\lambda=0)} = f(u) = \int_0^u r\,du = \int_0^\varphi \rho\,d\varphi$

where:

$$f(u) = l_\varphi = A\varphi - B\sin 2\varphi + C\sin 4\varphi - D\sin 6\varphi + \ldots$$

$$f^{I}(u) = r = N\cos\varphi$$

$$f^{II}(u) = \frac{dr}{du} = \frac{dr}{d\varphi}\frac{d\varphi}{du} = -r\sin\varphi = -N\sin\varphi\cos\varphi$$

$$etc.$$

- Map projection equations:

$$y = l_\varphi + \frac{1}{2}\lambda^2 N\sin\varphi\cos\varphi + \frac{1}{24}\lambda^4 N\sin\varphi\cos^3\varphi\left(5 - t^2 + 9\eta^2 + 4\eta^4\right)$$

$$x = \lambda N\cos\varphi + \frac{1}{6}\lambda^3 N\cos^3\varphi\left(1 - t^2 + \eta^2\right) +$$

$$+\frac{1}{120}\lambda^5 N\cos^5\varphi\left(5 - 18t^2 + t^4 + 14\eta^2 - 58t^2\mu^2\right)$$

being: $t = \tan\varphi$

$$\eta^2 = \frac{N-\rho}{\rho} = \frac{a^2 - c^2}{c^2}\cos^2\varphi$$

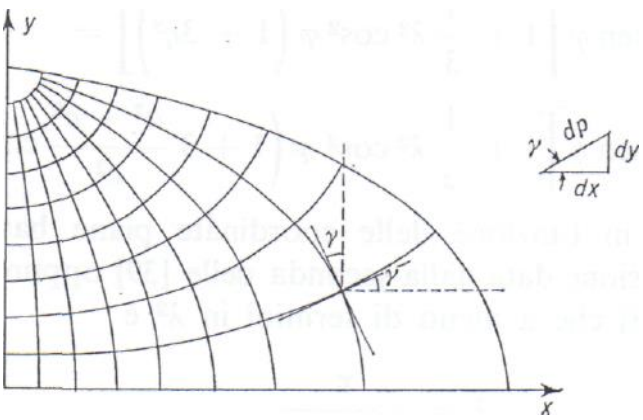
- Map projection reverse equations: $y + ix = f(u + i\lambda) \quad \Rightarrow \quad u + i\lambda = g(y + ix)$

$$\varphi = \varphi_1 - \frac{x^2}{2N_1^2}\left(1 + \eta_1^2\right)t_1 + \frac{x^4}{24N_1^4}\left(5 + 3t_1^2 + 6\eta_1^2 - 6\eta_1^2 t_1^2\right)t_1$$

$$\lambda = \frac{x}{N_1\cos\varphi_1} - \frac{x^3}{6N_1^3\cos\varphi_1}\left(1 + 2t_1^2 + \eta_1^2\right) + \frac{x^5}{120N_1^5\cos\varphi_1}\left(5 + 28t_1^2 + 24t_1^4 + 6\eta_1^2 + 8\eta_1^2 t_1^2\right)$$

being: $t_1 = \tan\varphi_1 \qquad \eta_1^2 = \left(\frac{N-\rho}{\rho}\right)_{\varphi=\varphi_1}$

<u>Meridian convergence</u>



$$tan\,\gamma = \frac{dy}{dx} = \frac{dy}{d\lambda}d\lambda \Big/ \frac{dx}{d\lambda}d\lambda = \frac{dy}{d\lambda}\Big/\frac{dx}{d\lambda} = \lambda\, sin\,\varphi\left(1+\frac{\lambda^2}{3}cos^2\,\varphi\left(1+t^2\right)+...\right) = \lambda\, sin\,\varphi + \frac{\lambda^3}{3}sin\,\varphi + ...$$

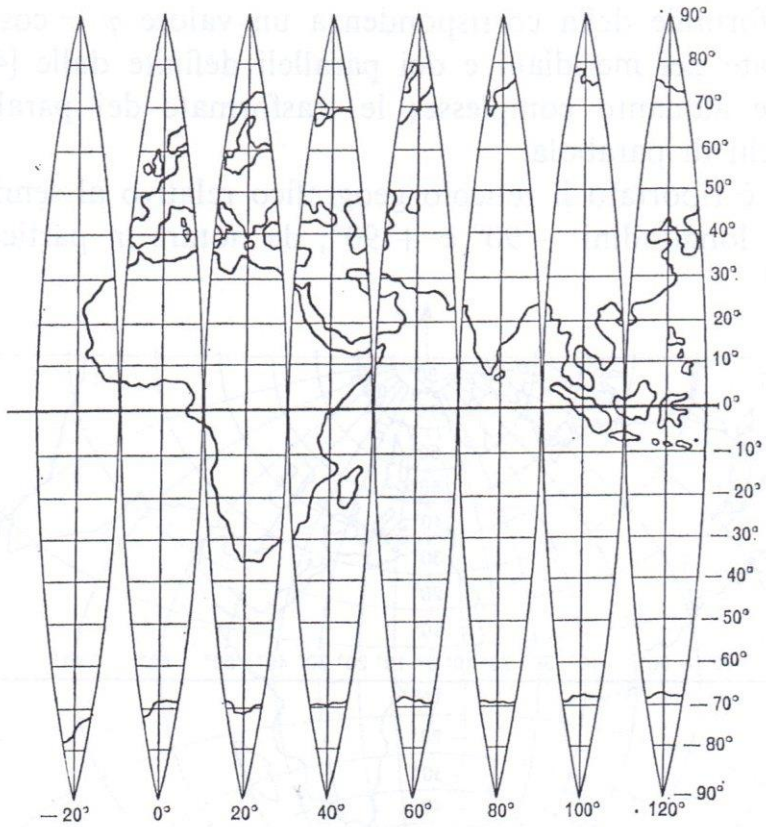
$$\gamma = \lambda\, sin\,\varphi\left(1+\frac{1}{3}\lambda^2\, cos^2\,\varphi\left(1+3\eta^2\right)\right)+... = \lambda\, sin\,\varphi\left(1+\frac{1}{3}\lambda^2\, cos^2\,\varphi\left(1+3\frac{a^2-c^2}{c^2}cos^2\,\varphi\right)\right)+... =$$

$$= arctan\left(\frac{x}{N}tan\,\varphi\left(1+\frac{1}{3}\frac{x^2}{N^2\, cos^2\,\varphi}+...\right)\right)$$

being: $$\frac{dy}{d\lambda} = \lambda N\, sin\,\varphi\, cos\,\varphi\left(1+\frac{\lambda^2}{6}cos^2\,\varphi\left(5-t^2\right)+...\right)$$

$$\frac{dx}{d\lambda} = N\, cos\,\varphi\left(1+\frac{\lambda^2}{6}cos^2\,\varphi\left(1-t^2\right)+...\right)$$

$$\gamma = tan\,\gamma - tan^3\,\gamma/3 + tan^5\,\gamma/5 - ...$$



<u>Linear deformation</u>

$$m^2 = g^* = g/r^2 = \left((\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2\right)/r^2 = 1+\lambda^2\, cos^2\,\varphi$$

$$m = \sqrt{1+\lambda^2\, cos^2\,\varphi} \cong 1+\frac{\lambda^2}{2}cos^2\,\varphi = 1+\frac{x^2}{2N^2}$$

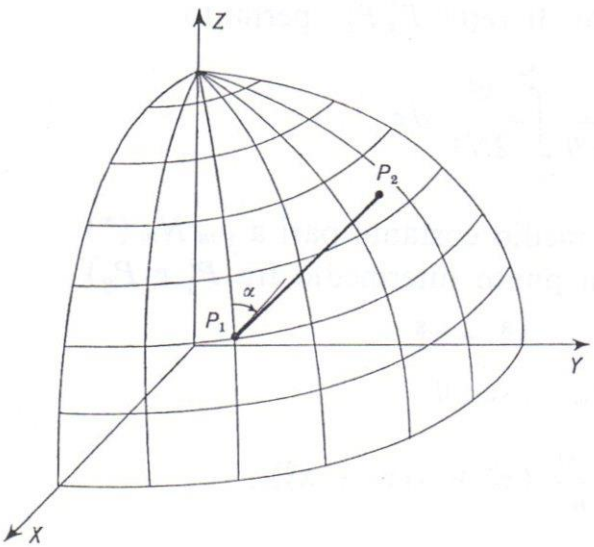
$$ds_e = \frac{ds_r}{m} = ds_r \Big/ \left(1 + \frac{x^2}{2N^2}\right) \cong ds_r \left(1 - \frac{x^2}{2N^2}\right) = \frac{dx}{\sin\vartheta} - \frac{1}{2\sin\vartheta}\frac{x^2}{N^2}dx$$

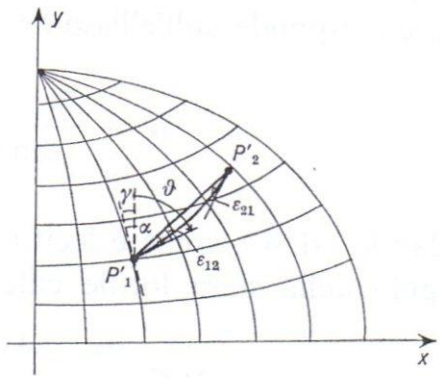
$$ds_r = \frac{dx}{\sin\vartheta}$$

being: $$s_e = \frac{x_2 - x_1}{\sin\vartheta} - \frac{1}{\sin\vartheta}\int_{x_1}^{x_2}\frac{x^2}{2N^2}dx = \frac{x_2 - x_1}{\sin\vartheta} - \frac{1}{6\rho_m N_m}\frac{x_2^3 - x_1^3}{\sin\vartheta} =$$

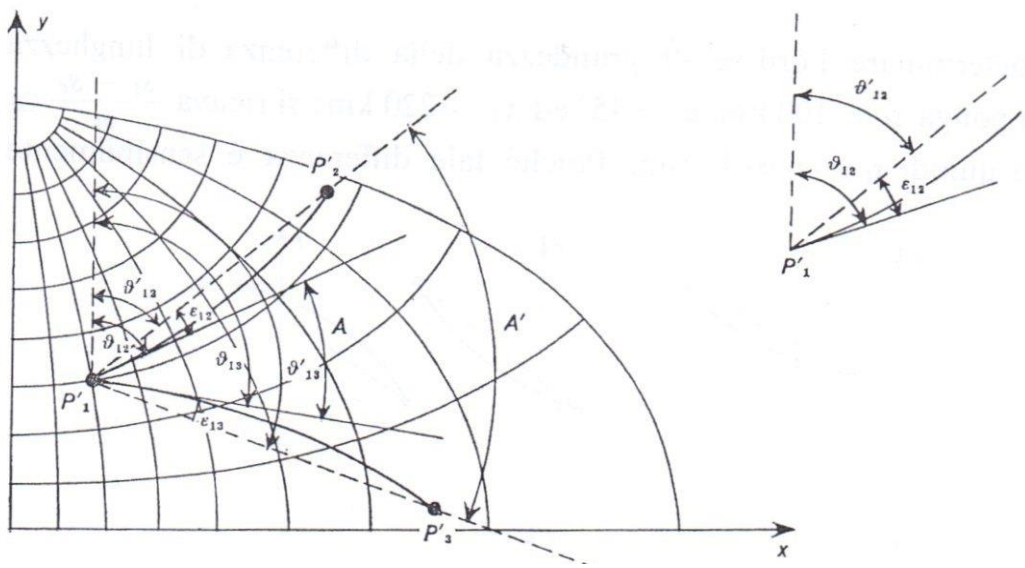
$$= \frac{x_2 - x_1}{\sin\vartheta} - \frac{1}{6\rho_m N_m}\frac{x_2 - x_1}{\sin\vartheta}\left(x_1^2 + x_1 x_2 + x_2^2\right) = s_r\left(1 - \frac{x_1^2 + x_1 x_2 + x_2^2}{6\rho_m N_m}\right)$$

$$m_{12} = \frac{s_r}{s_e} = 1 + \frac{x_1^2 + x_1 x_2 + x_2^2}{6\rho_m N_m}$$ (in the segment $1,2$)





$$s_t \cong s_c$$



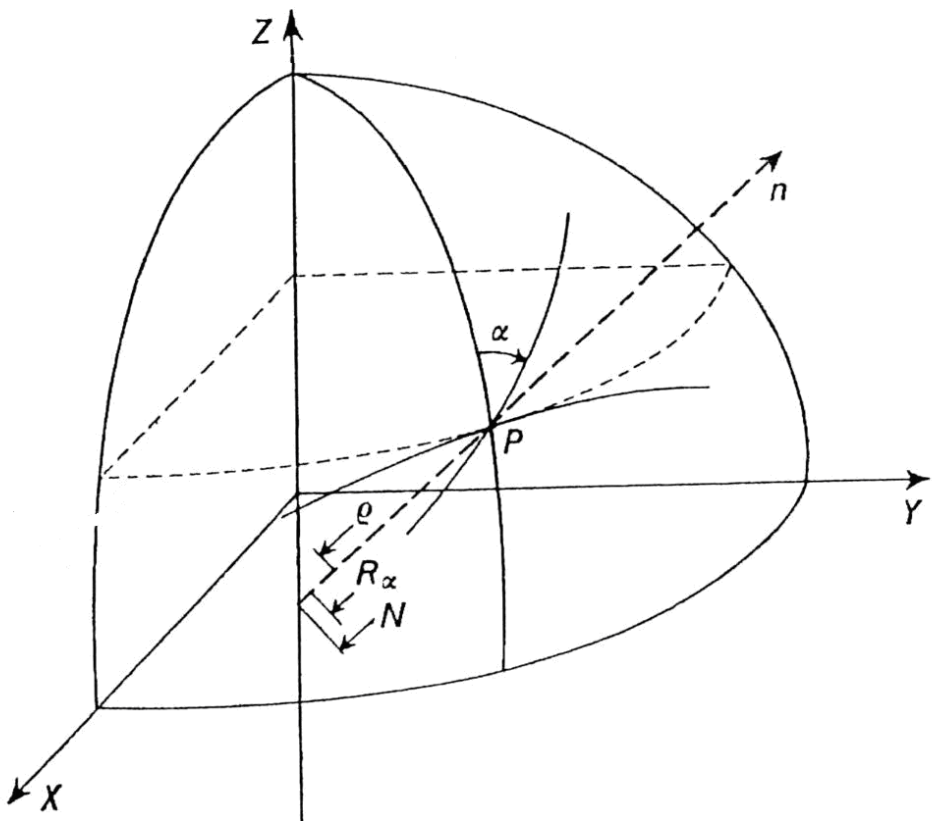
$$\frac{s_t - s_c}{s_t} = \frac{x_1(x_1 + s_t \sin\alpha) s_t^2 \cos^2\alpha}{24N^4}$$

$$\varepsilon_{12} = \frac{1}{6\rho_c N_c}(y_1 - y_2)(2x_1 + x_2)$$

$$\varepsilon_{21} = \frac{1}{6\rho_c N_c}(y_2 - y_1)(2x_2 + x_1)$$

<u>Ellipsoid geometry</u>

BESSEL (1841)	$a = 6.377.397\ m$	$\alpha = 1/299.2$
HAYFORD(1909)	$a = 6.378.388\ m$	$\alpha = 1/297.0$
WGS84 (1984)	$a = 6.378.137{,}0\ m$	$\alpha = 1/298{,}257223563$



- Ellipsoid principal axes: $\rho = \dfrac{a(1-e^2)}{\sqrt{(1-e^2 \sin^2\varphi)^3}}$ being: $e^2 = \alpha(2-\alpha)$

$$N = \frac{a(1-e^2)}{\sqrt{1-e^2 \sin^2\varphi}}$$

- Local sphere radius: $R = \sqrt{\rho N} = \dfrac{a(1-e^2)}{1-e^2 \sin^2\varphi}$

PART IV – A SUMMARY OF THE MOST IMPORTANT HISTORICAL PASSAGES

The maps of Western culture

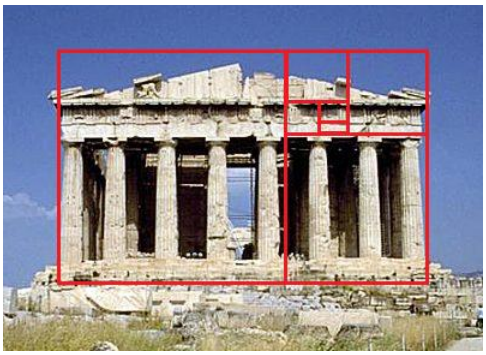
The birth of thought and Western civilization in Mesopotamia and Egypt

Development of Western thought and civilization from classical Greece to Hellenistic world

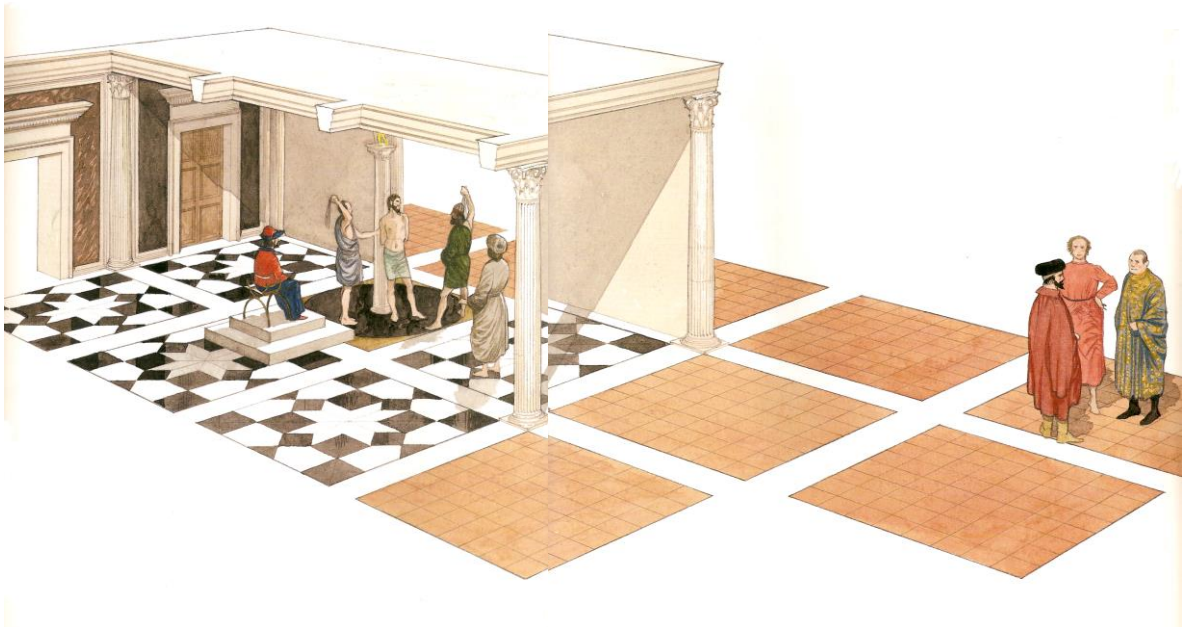
The apogee of the ancient world from the late Roman Republic to the Roman Empire

The high Middle Ages between Byzantium and Islam in North Africa
and Southwest Europe (mainly Spain and Sicily)

A path of space and time, from Ancient World to Middle Age and Renaissance, through art, architecture, music, literature, philosophy, physics and mathematics, shows the passage from a circular time to a linear one, and from a space based on the Euclidean geometry to an other one founded on the perspective.



The Parthenon (Athens) and the “golden mean”



Piero della Francesca, Scourging of Chrtist (Galleria Nazionale delle Marche, Urbino)
and its “solid” reconstruction by means of a geometrically rigorous perspective

The Middle Ages of the free Italian municipalities, the Italian maritime republics
and the Hanseatic League (in the North Sea and the Baltic Sea)

Europe between Humanism (which developed from Italy and Flanders,
spreading slowly in the rest of Europe)
and the Renaissance (with a first significant presence of Northern Europe)

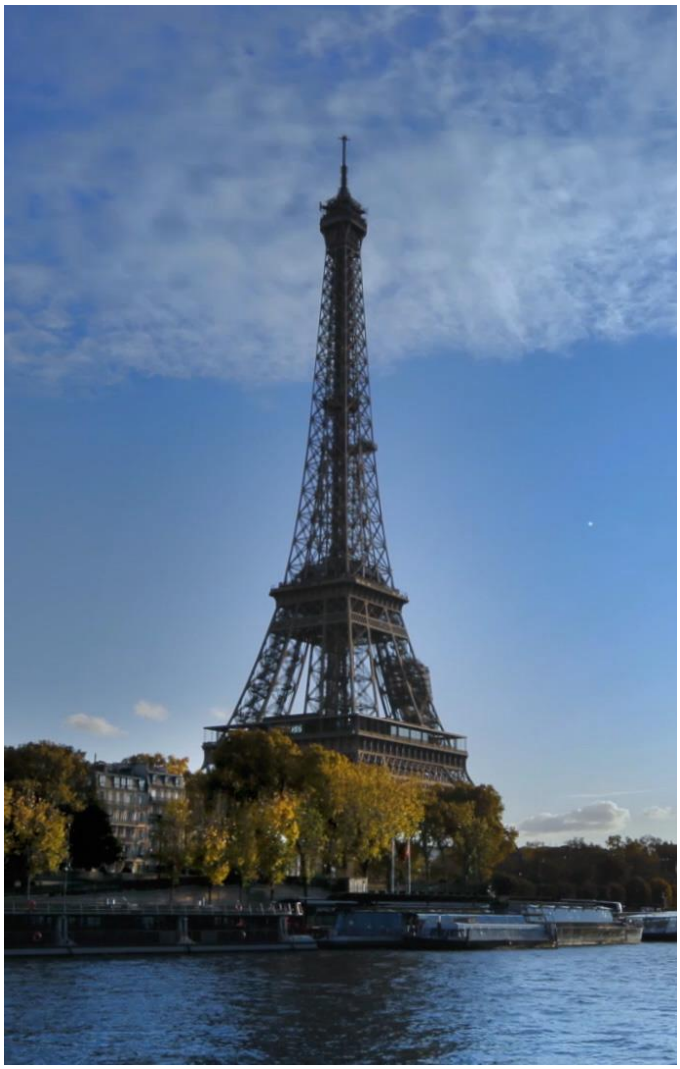
The European division between the Protestant Reformation and the Catholic Counter-Reformation, where France and Great Britain are in between
(while Eastern Europe is already separated from the Orthodox schism)

The beginnings of the Enlightenment (mainly France and Great Britain)

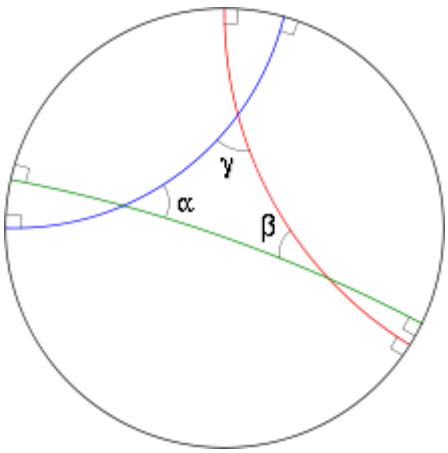
Europe in the Age of Enlightenment (with a first significant presence of Russia),
but after, French Revolution, the Napoleonic adventure and the national Resurgences

The conquest of freedom, in equity, in the United Europe, once again active in the Mediterranean basin

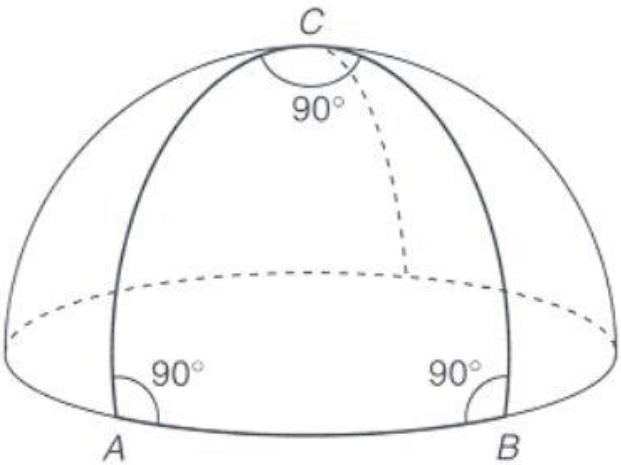
Nowadays the path of space and time, after the Golden siècle, the Enlightenment age, and the 19th and 20th centuries, shows a time part of the relativistic physics, and a space opened to the non-Euclidean geometries, where the innovations both in mathematics and physics not only involve other sciences and technologies, but also have influence on the modern development of art, architecture, music, literature and philosophy.



The Tour Eiffel (Paris) with its hyperbolic profile



Hyperbolic triangle



Elliptic triangle

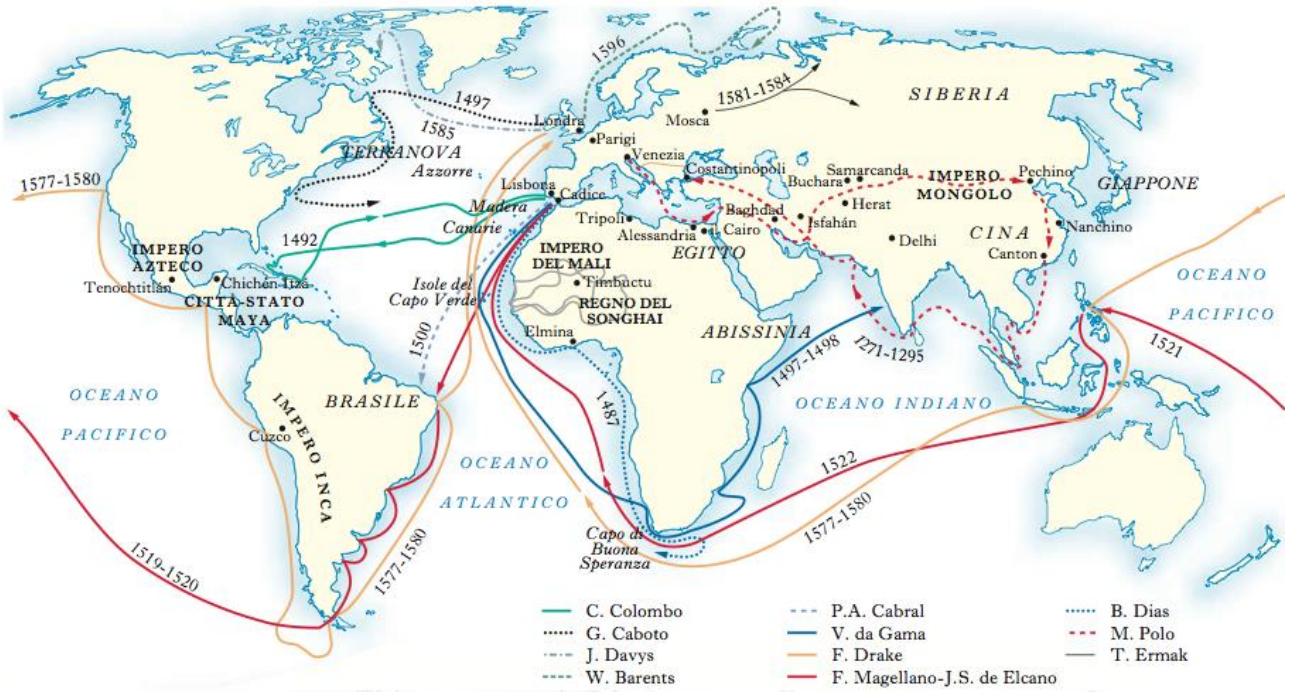
The Europe of the Iron Curtain and the opposition in the east-west blocks,
after two European and world wars, and the age of totalitarianism
(with a first significant presence of USA, after their independence)

The alarming current European and Mediterranean fragmentation in several opposing interest groups
(United Europa, Russia, Arab world, Israel, etc.)
in a world that has become global and multi-polar (Japan, China, India, South Africa, Latin America, etc.)

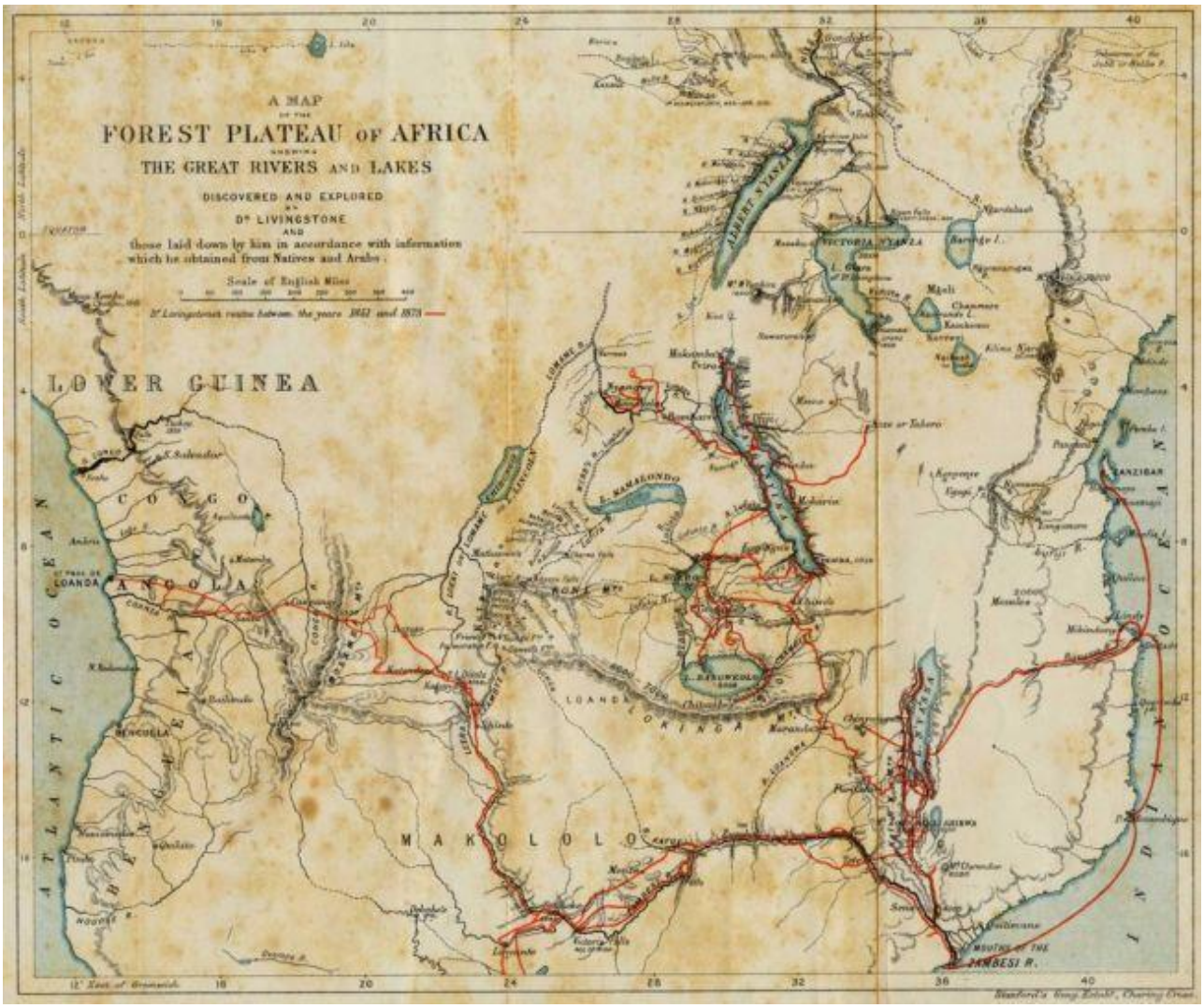
A brief history of Survey and Mapping from old antiquity to present contemporary age

Shield of Achilles

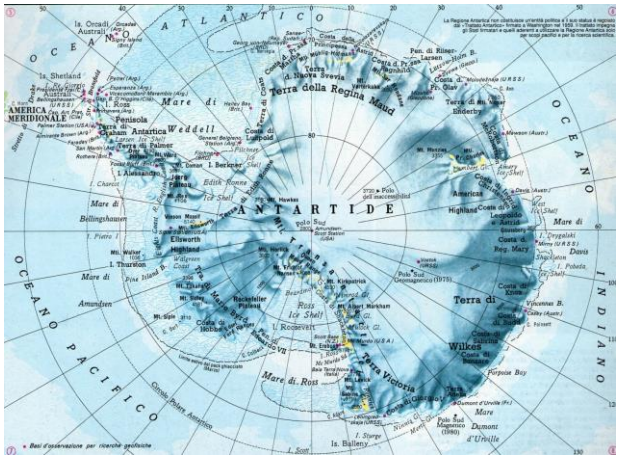
Hereford's map



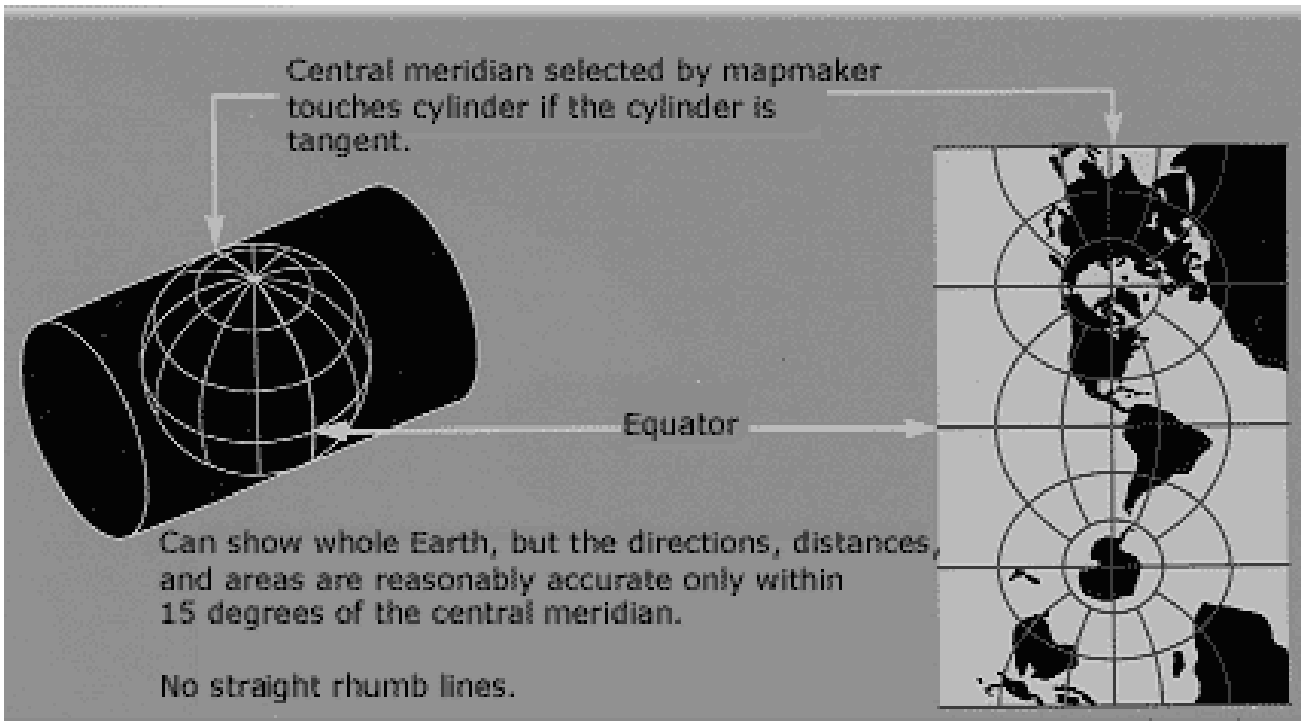
Medieval and Renaissance travels versus Asia and America



Travels in Africa of Livingstone and Stanley (in the 19th century)



The last explorations: North pole [15] South pole [16]

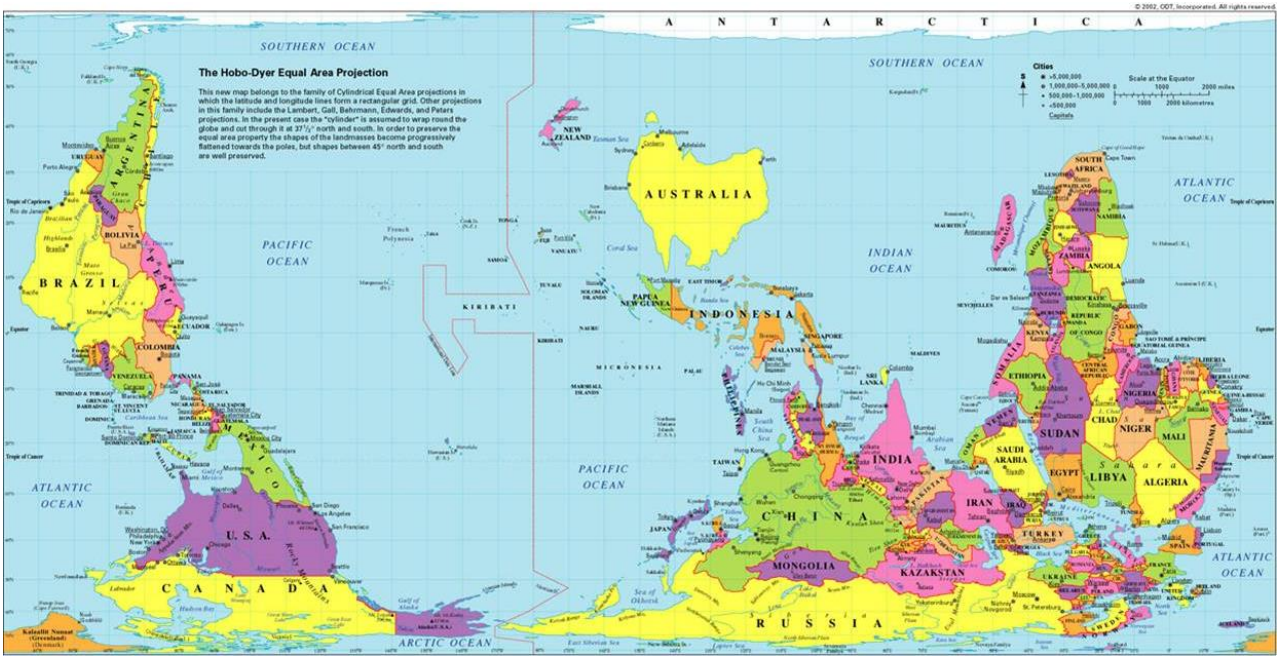


Universal Transverse Mercator (UTM)

Flag of the United Nations (with the UPS projection)

[15] After some attempts, the first official expedition was done by the Italian explorer: Nobile.
[16] The South pole were reached by the Norwegian explorer: Amundsen, and the English explorer: Scott (who died during the return).

Naïve maps spontaneously done by refugee–children of Darfur

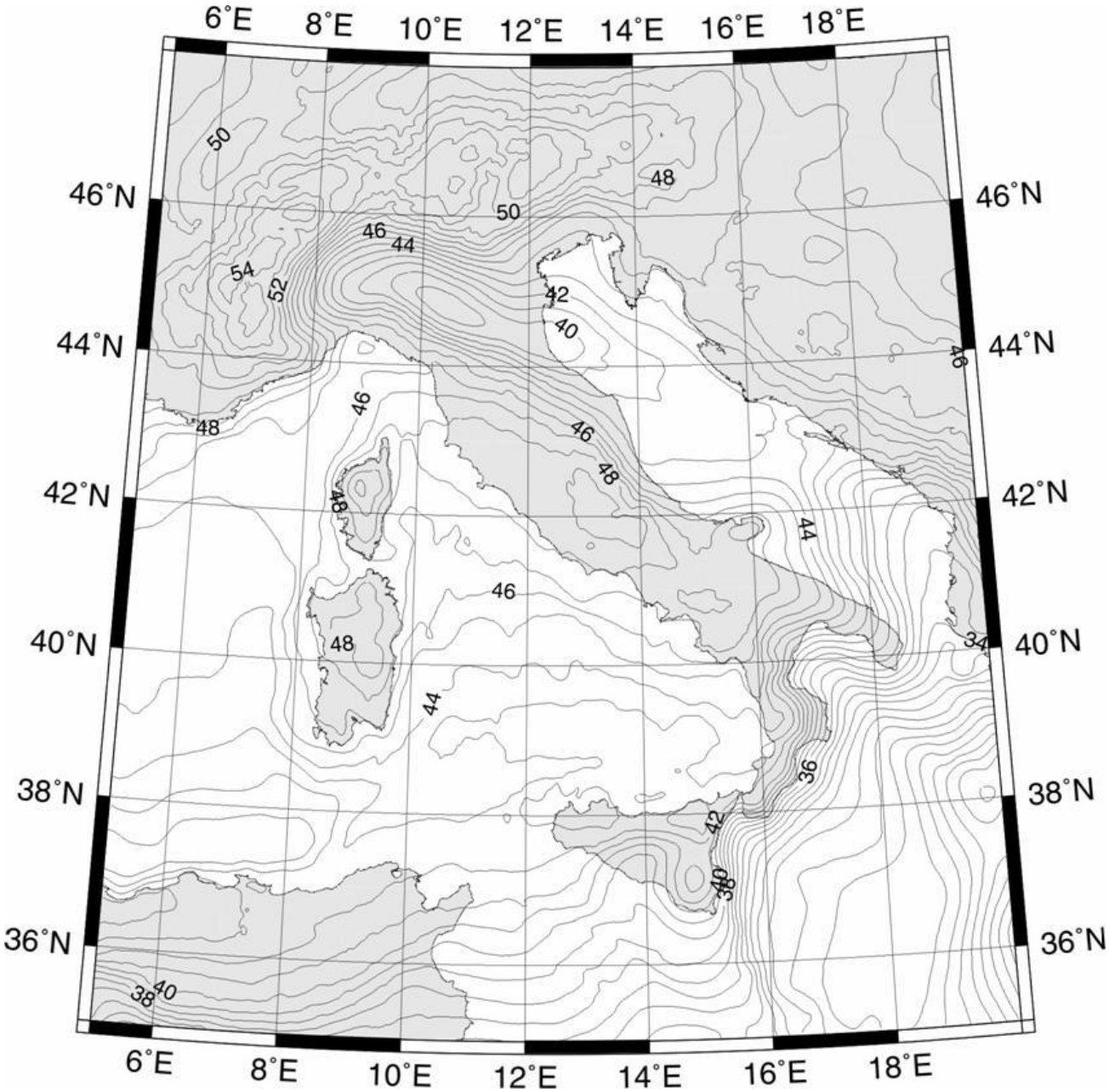


South–North planisphere

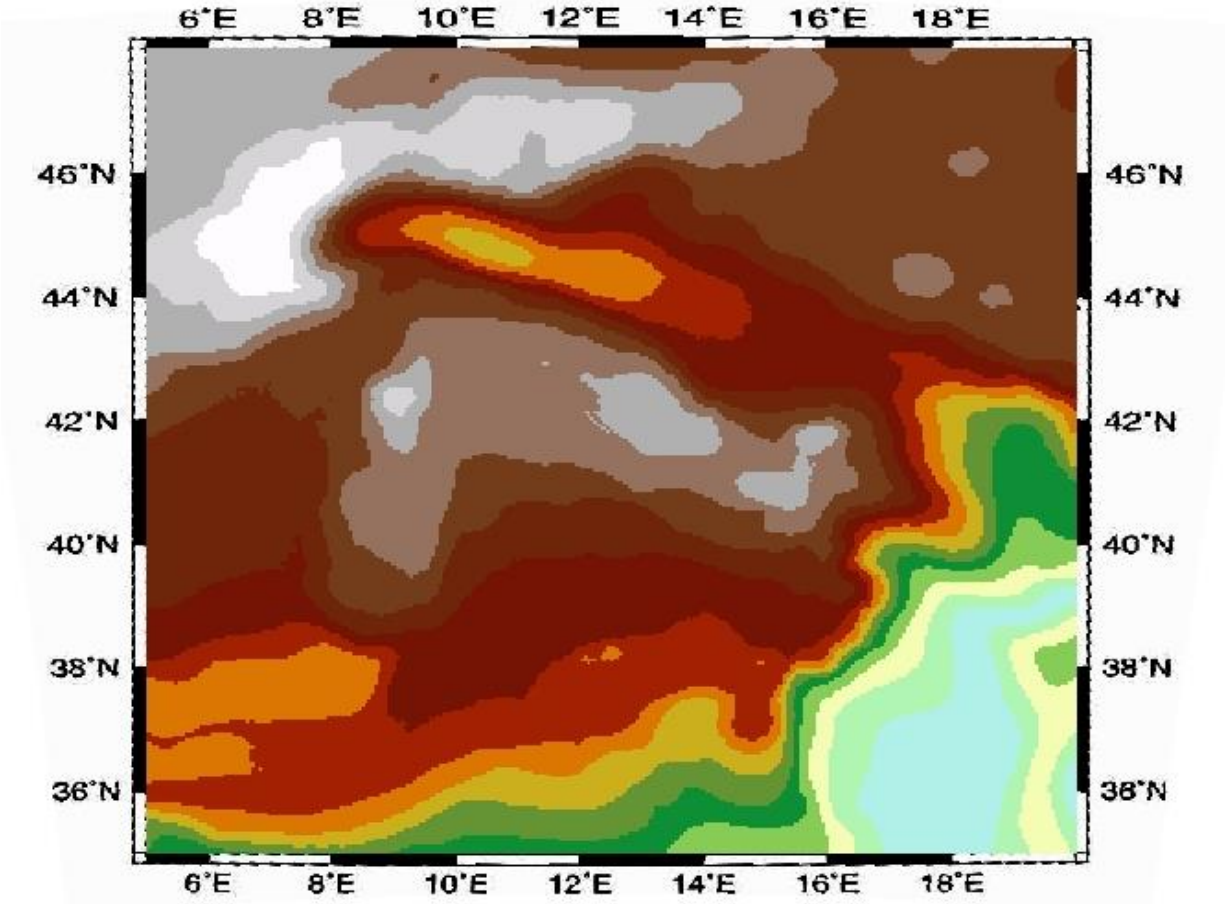
PART V – RECENT WORKS AT THE MILAN SCHOOL

Principal themes:

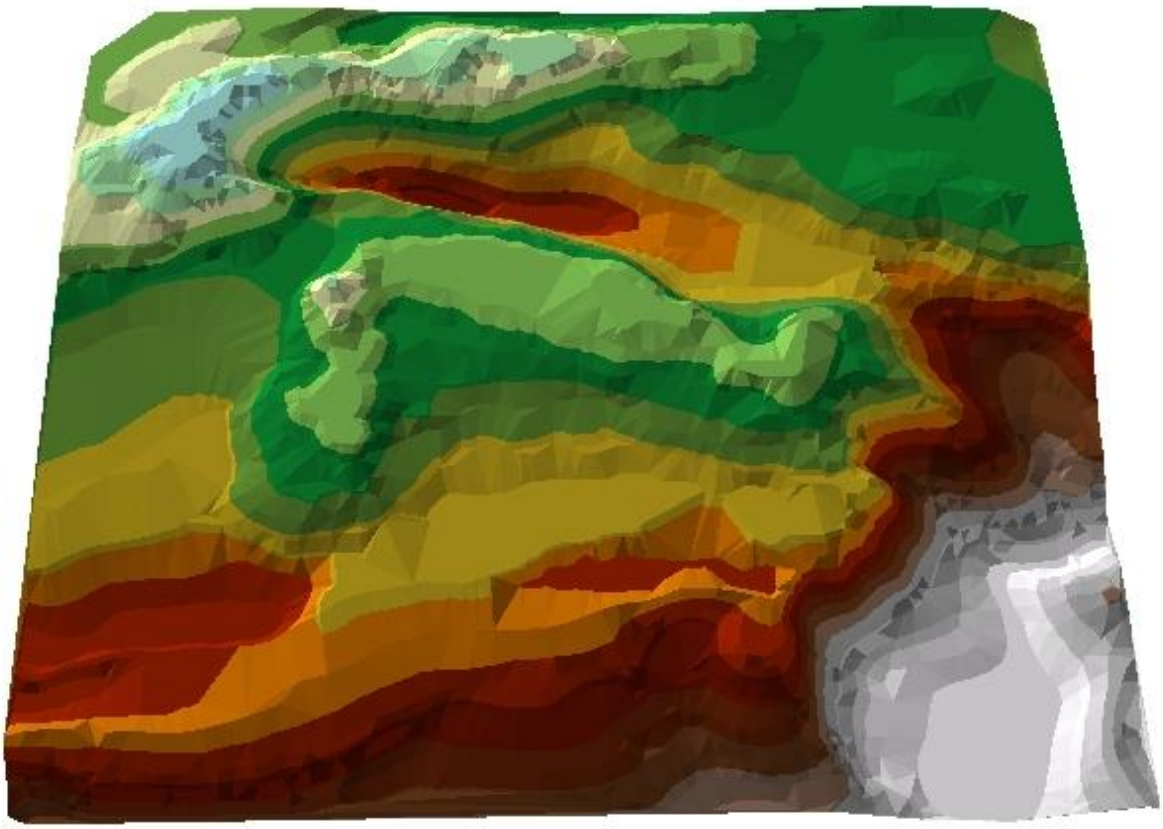
- Data processing;
- Survey and mapping:
  - Geoid computation;
  - Deformation monitoring by photogrammetry and applying geodesy;
  - GIS and thematic maps;
  - Architectural photogrammetry and surveying.
- Applied Geomatics.



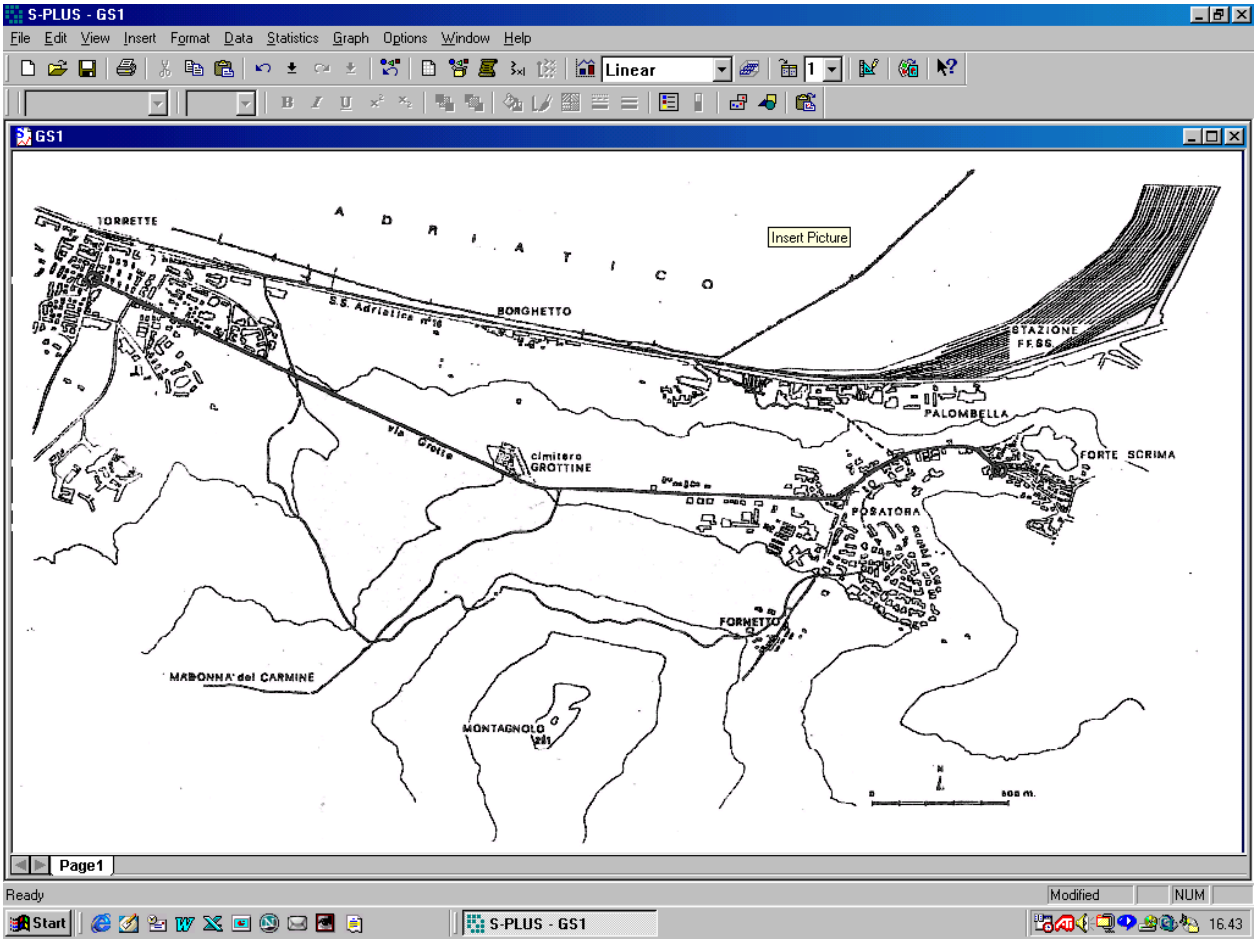
Contour line map of the Italian geoid



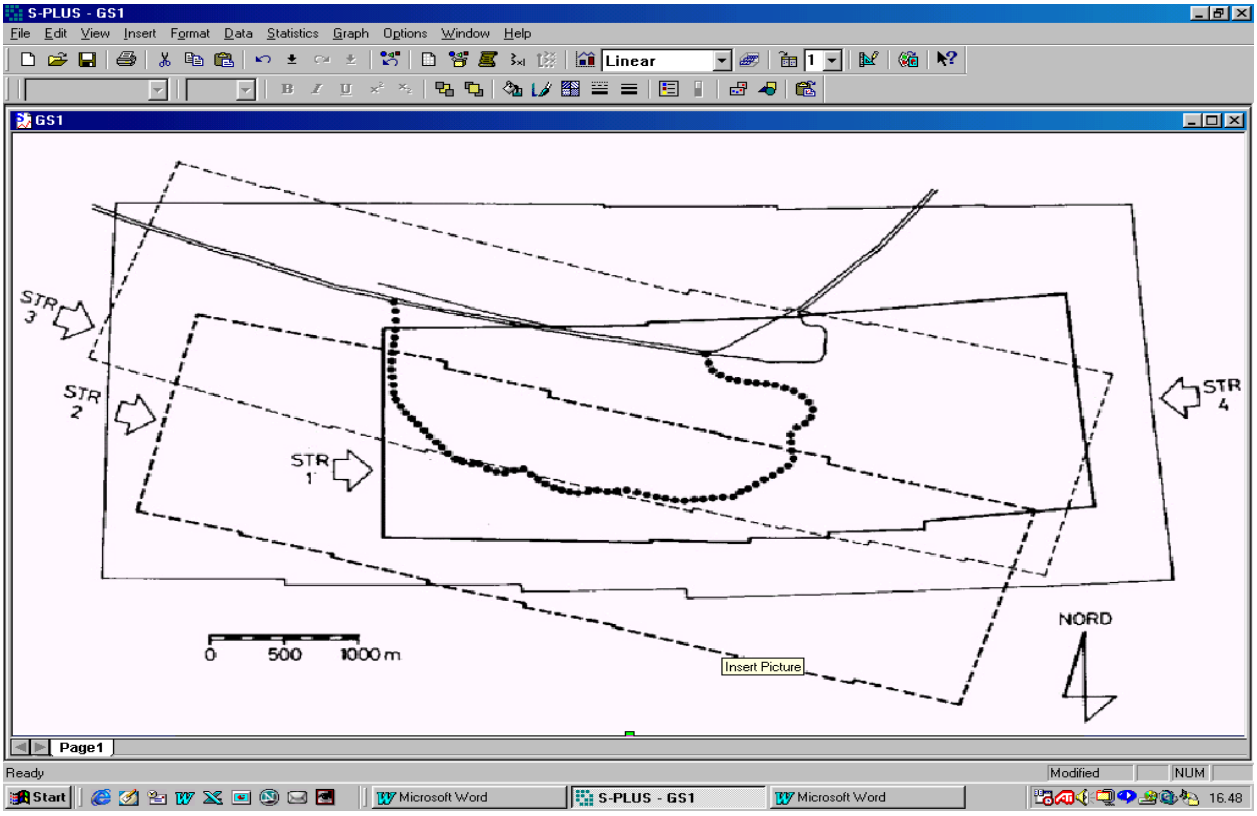
Color chart of the Italian geoid



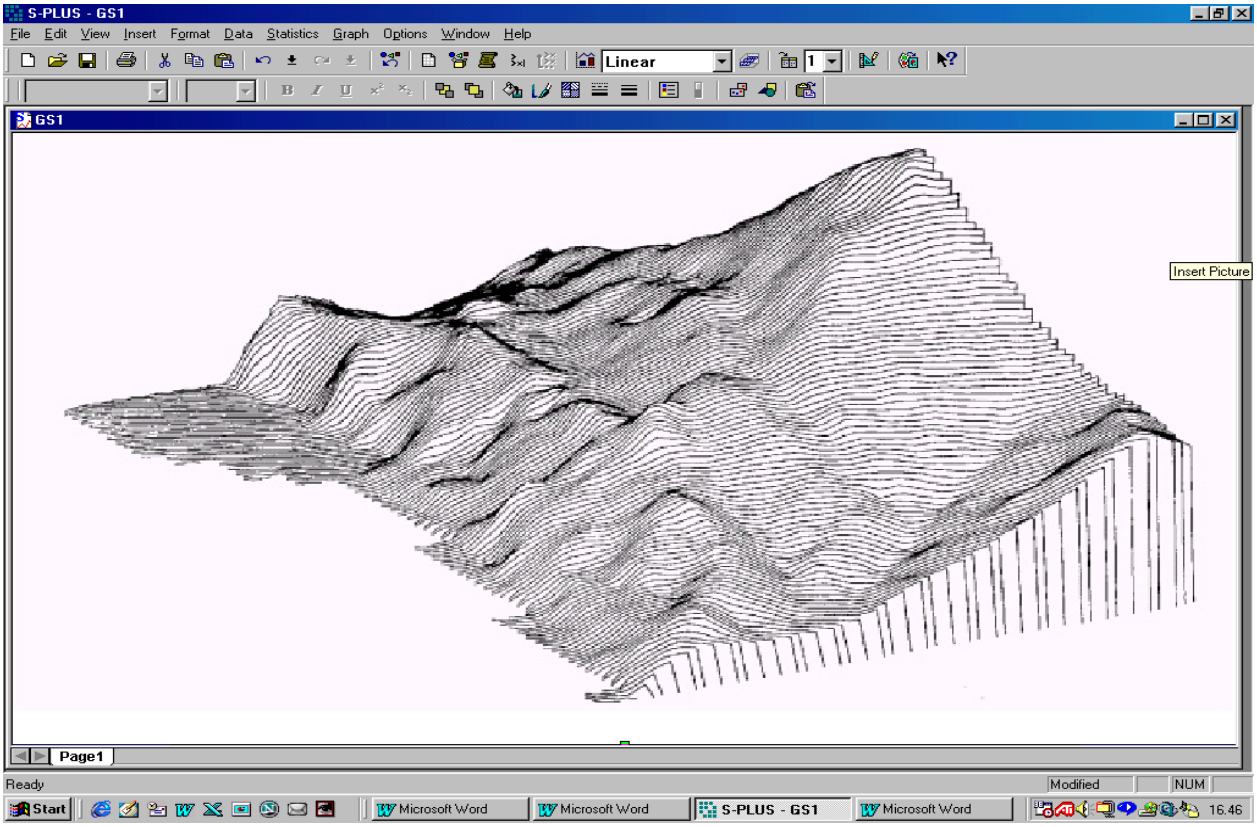
3D TIN of the Italian geoid



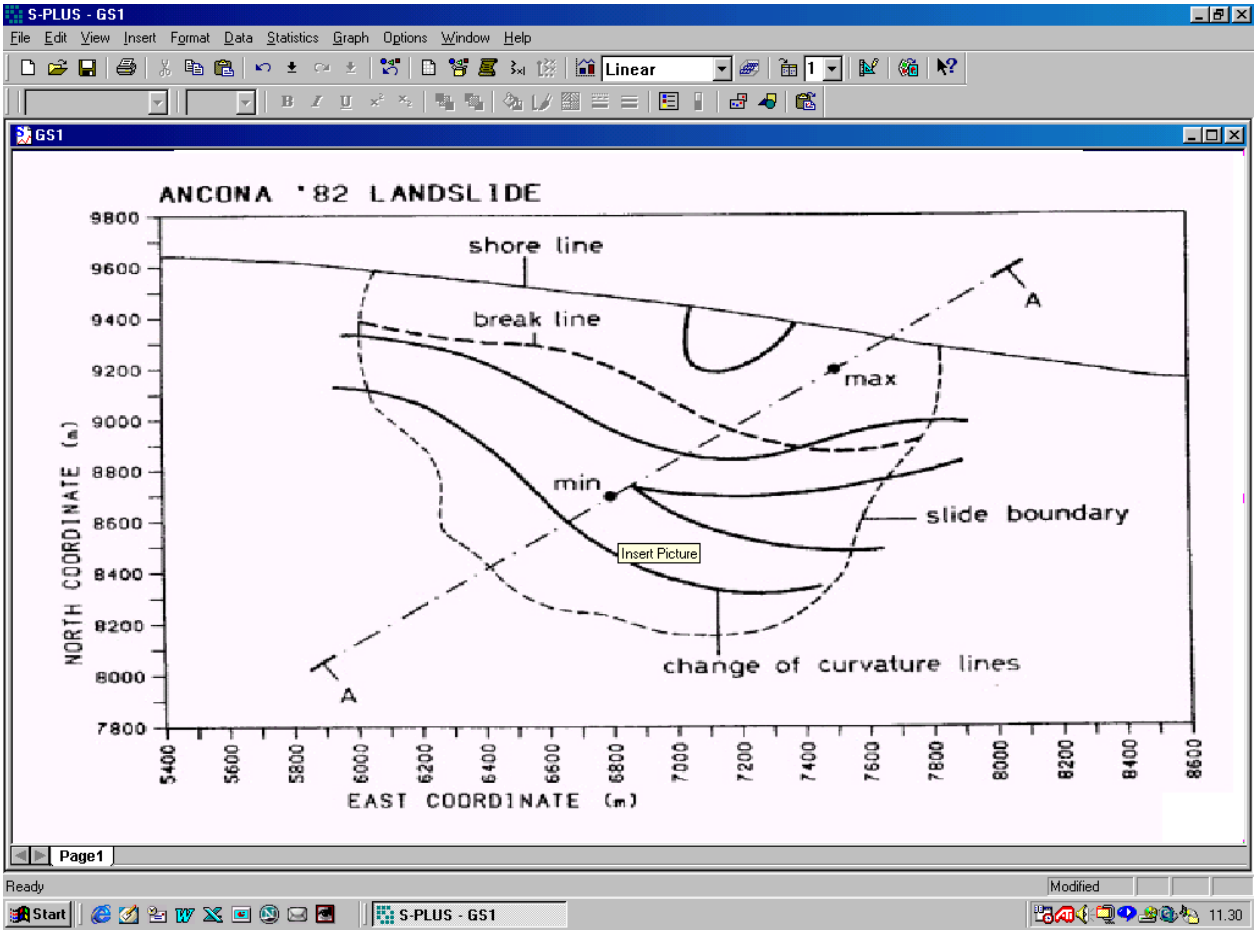
Map of the Ancona'82 Landslide



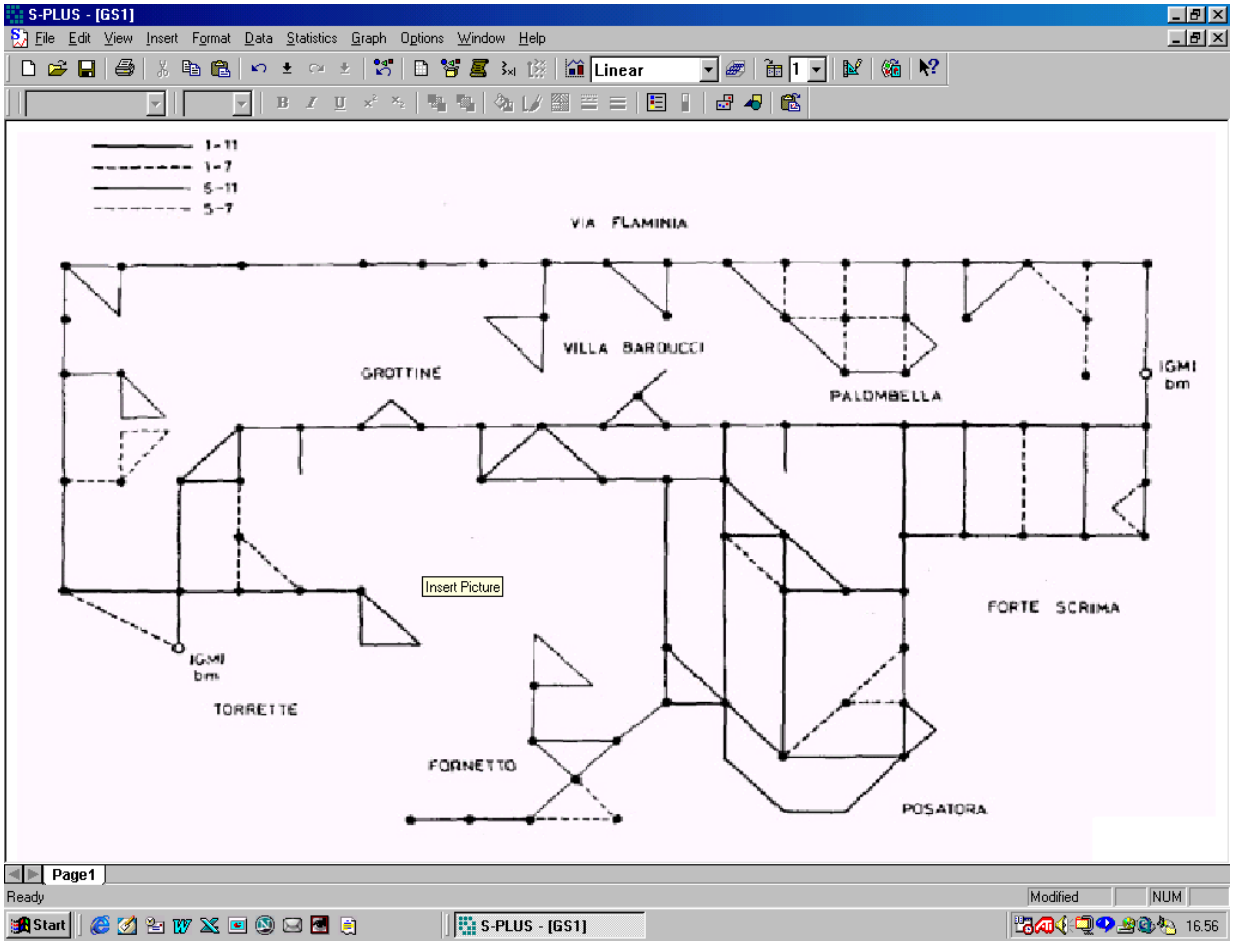
Photogrammetric Images before and after Ancona'82 Landslide



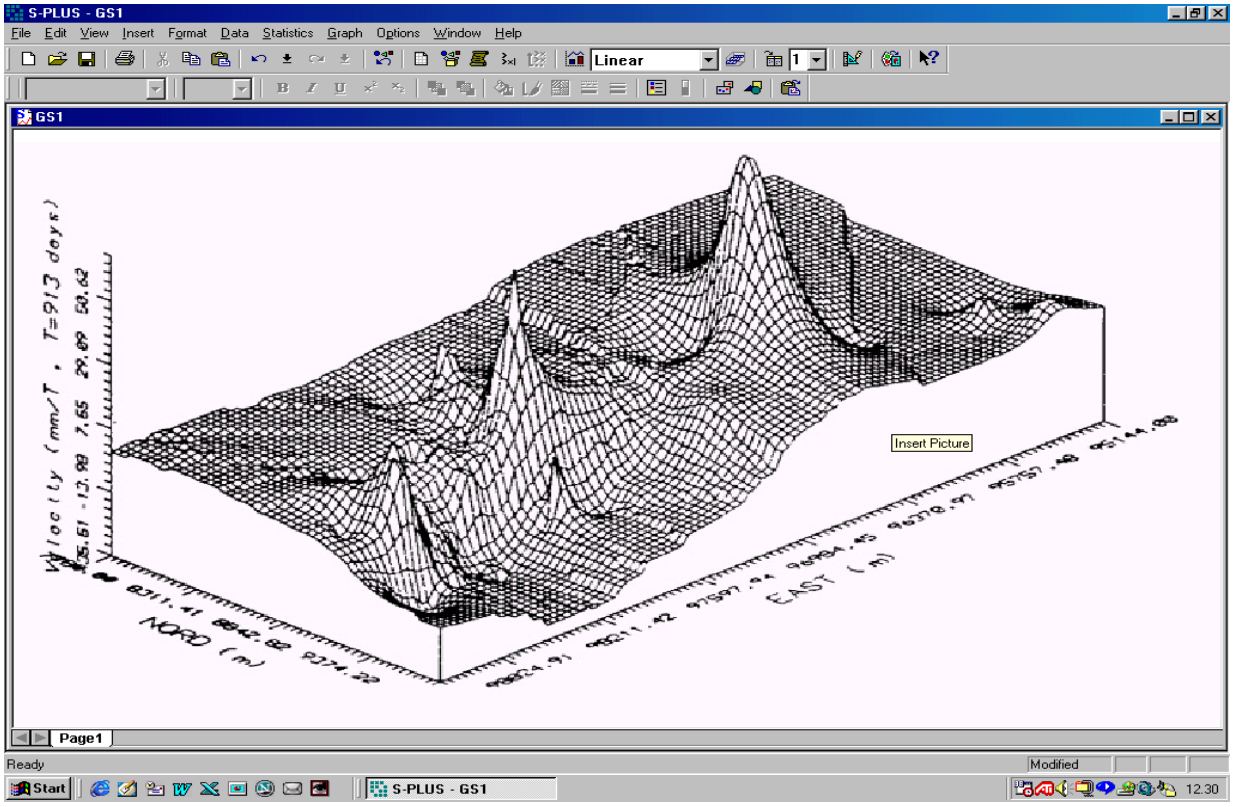
Model of Deformation Area of Ancona'82 Landslide



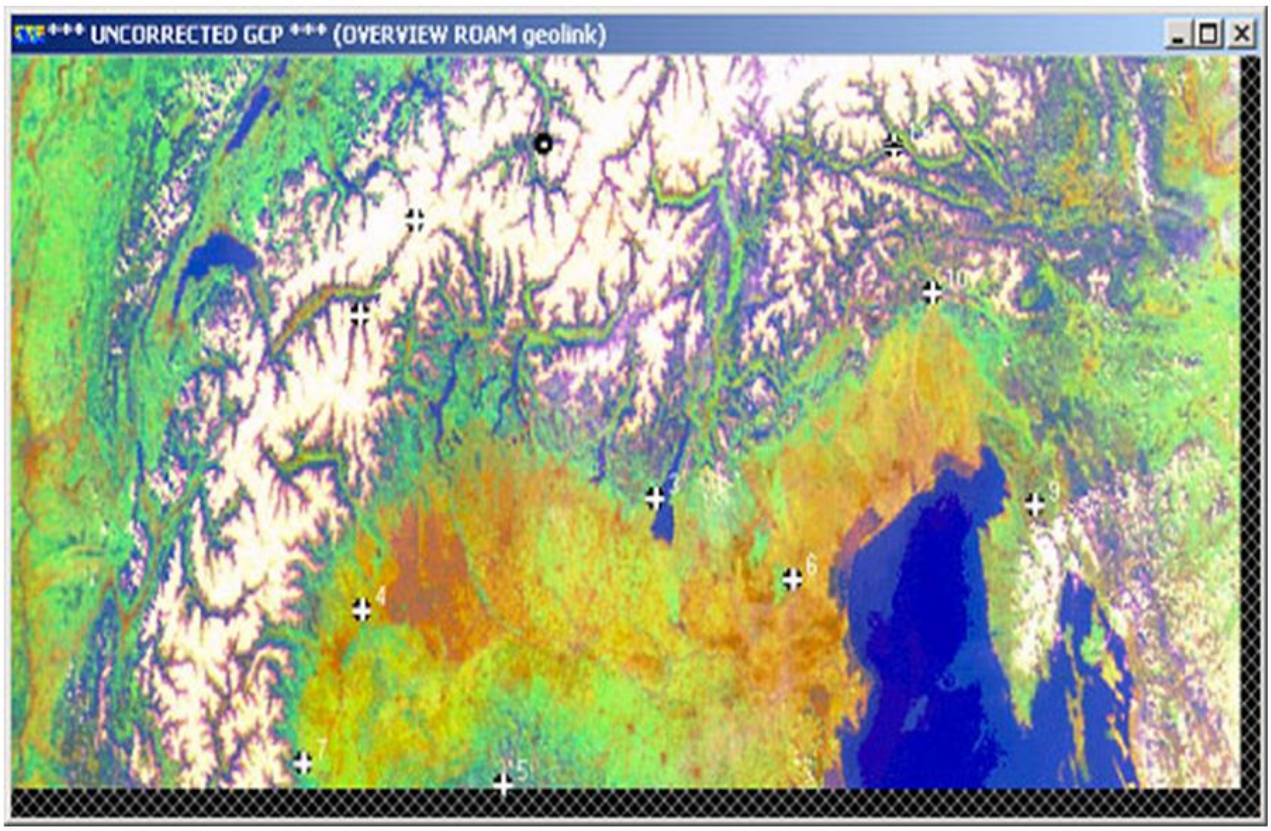
Features of Ancona'82 Landslide



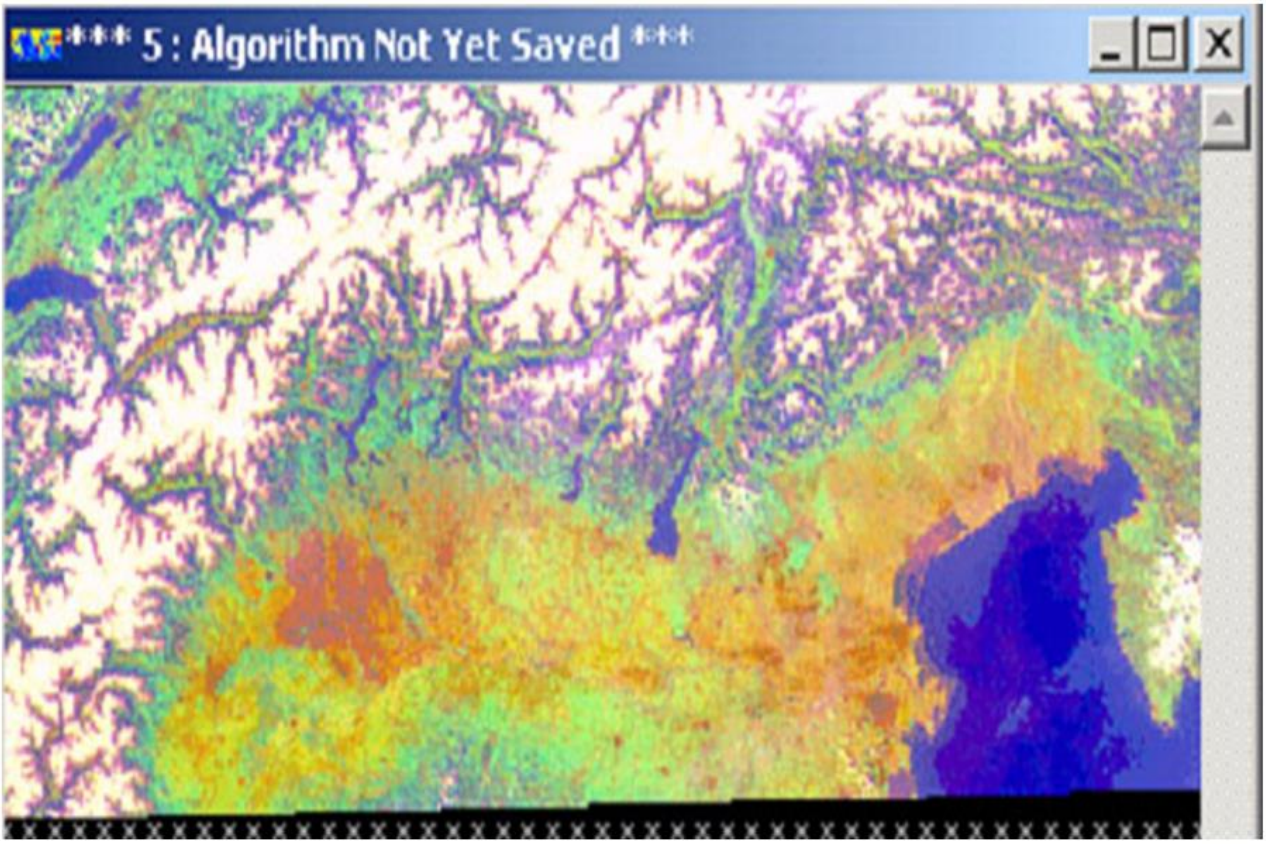
Geometric Leveling Network of Ancona'82 Landslide



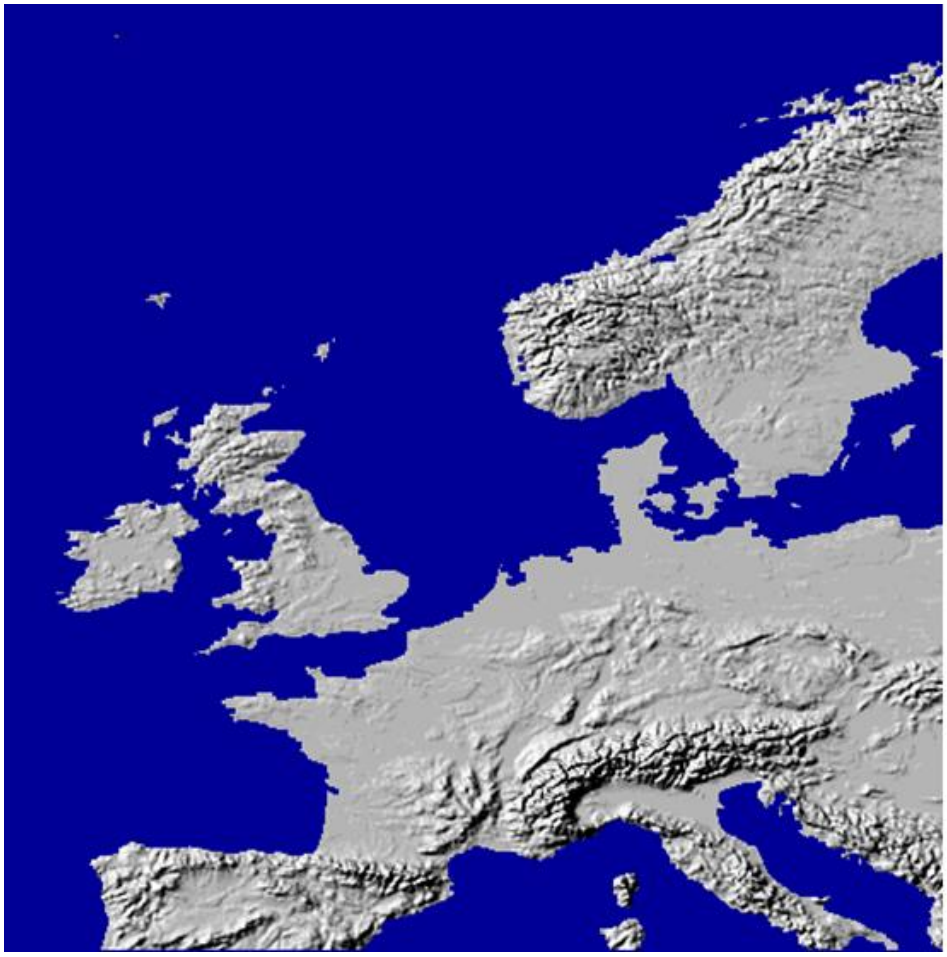
Model of Settlement Movement of Ancona'82 Landslide



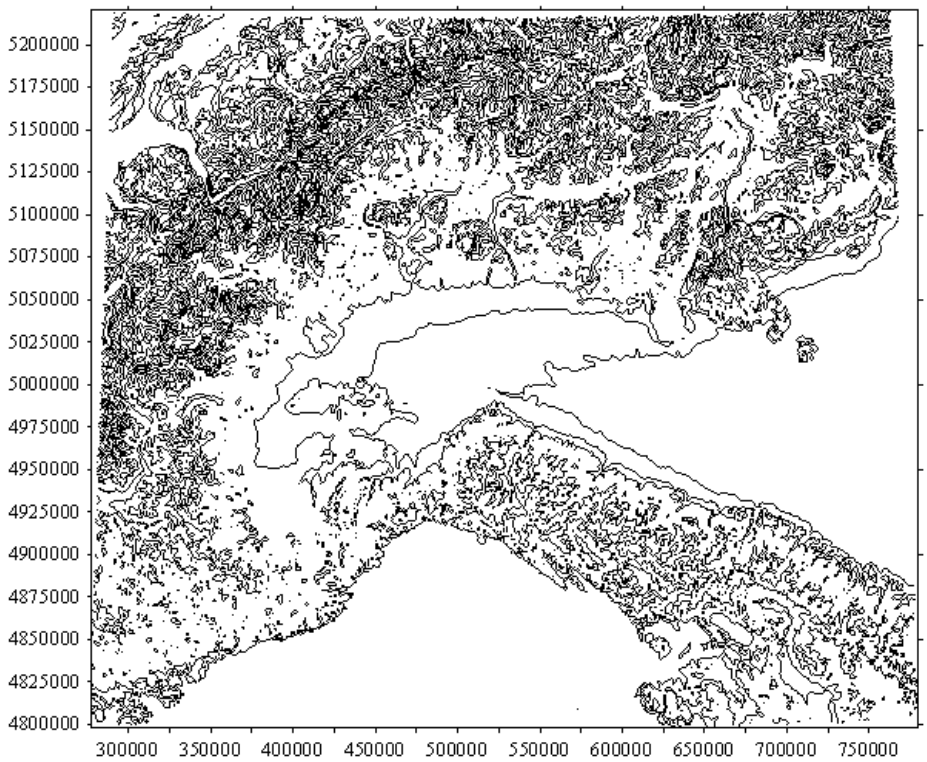
NOAA Image of Po Valley



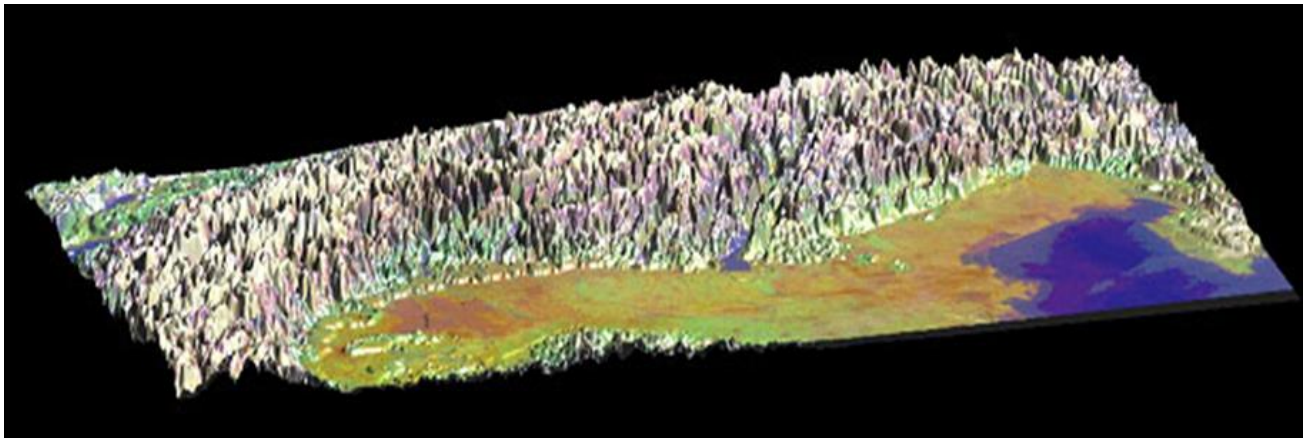
Geo-referenced NOAA Image of Po Valley



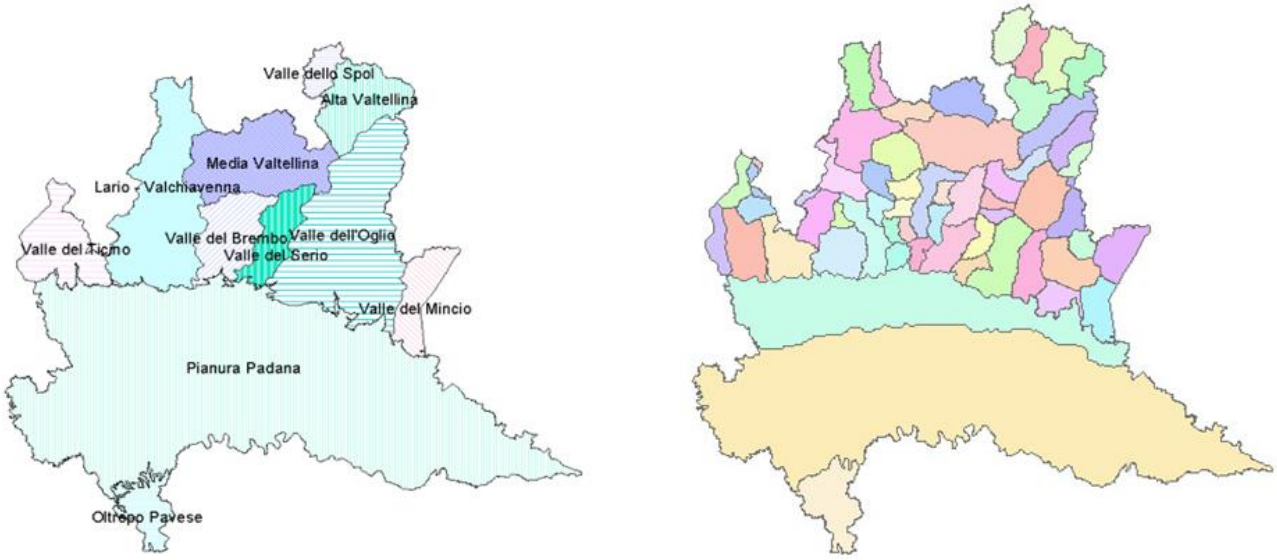
Shaded Relief Image of Western Europe



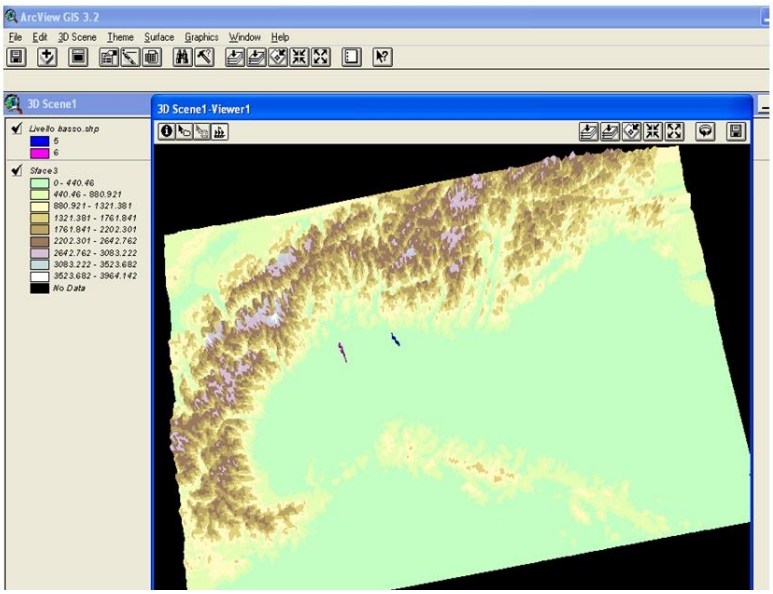
Contour Lines of Po Valley



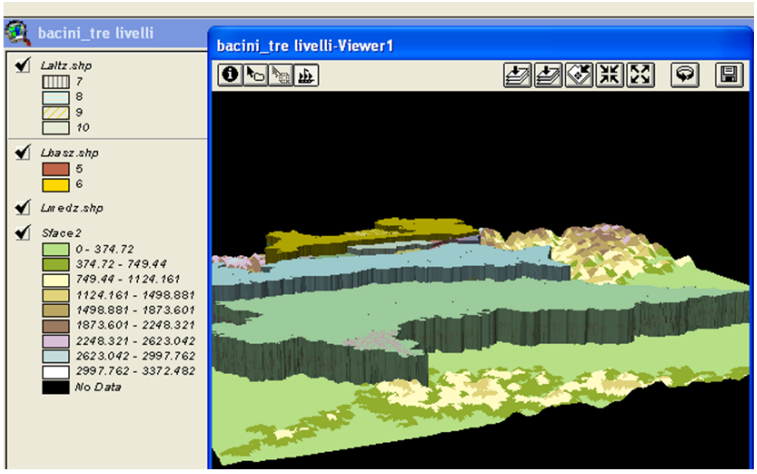
DEM with Texture Mapping of Geo-referenced NOAA Image of Po Valley



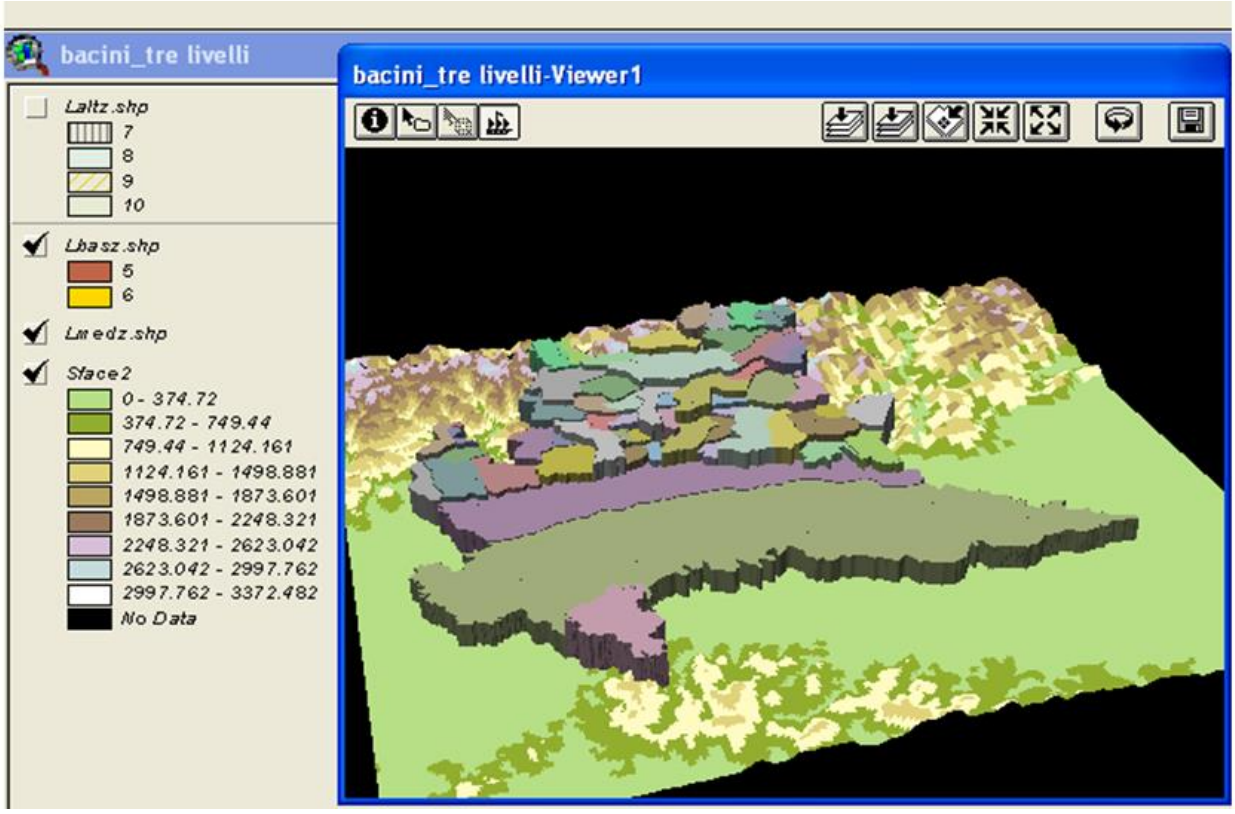
Administrative Basins and Air-basins between Ground Floor and 2,500 m. of Lombardy



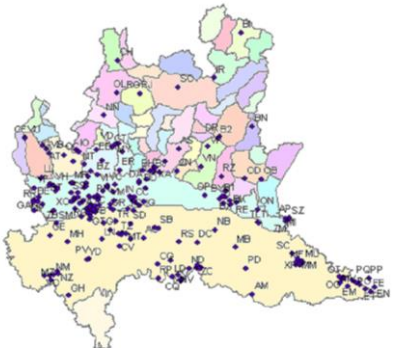
Canyon of Rivers Ticino and Adda



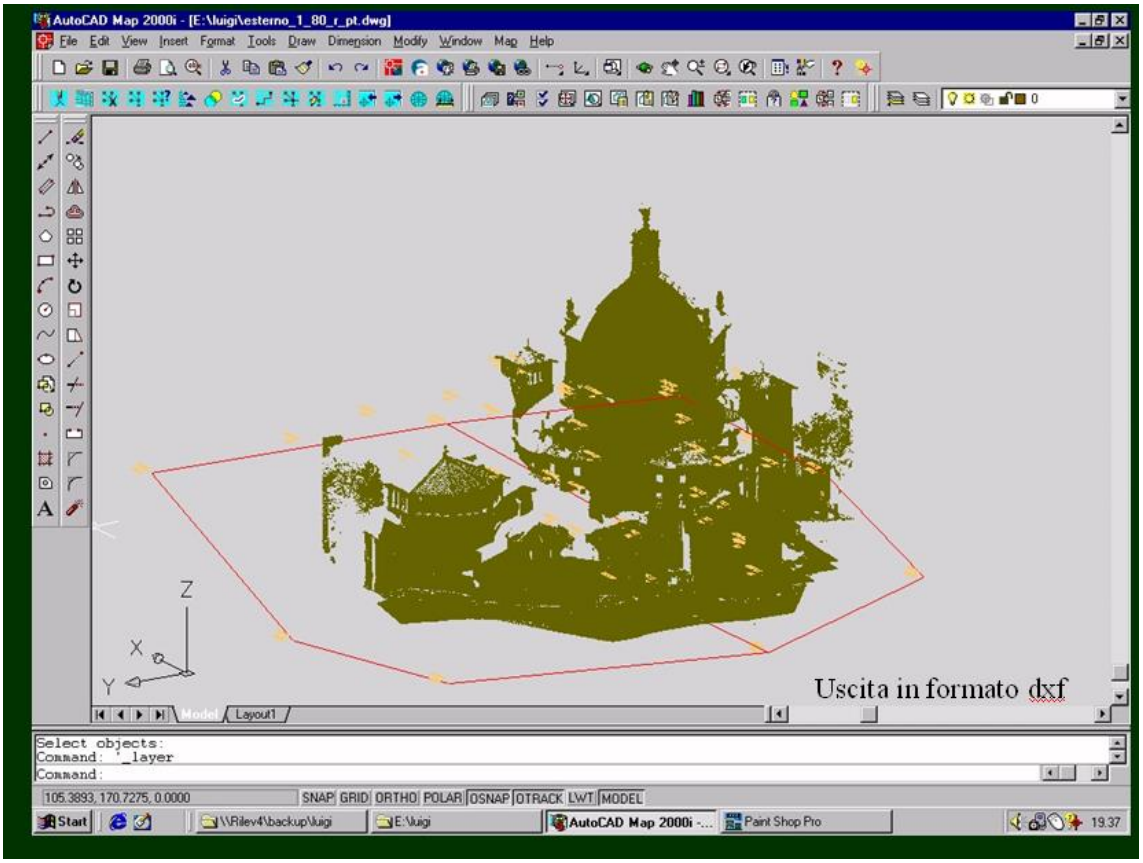
Air-fields of Orobie and Retiche Alps (over 2,500 m.)



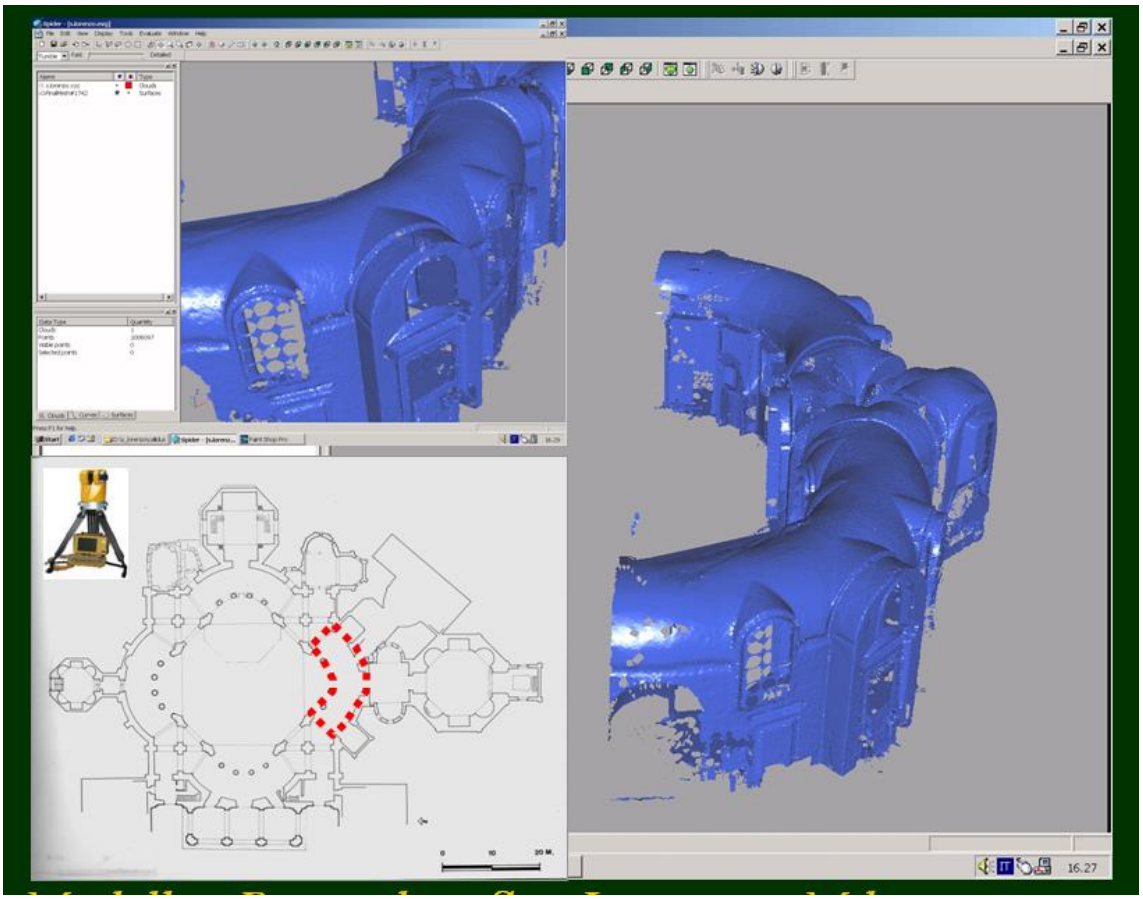
Air-basins between Ground Floor and 2,500 m. of Lombardy



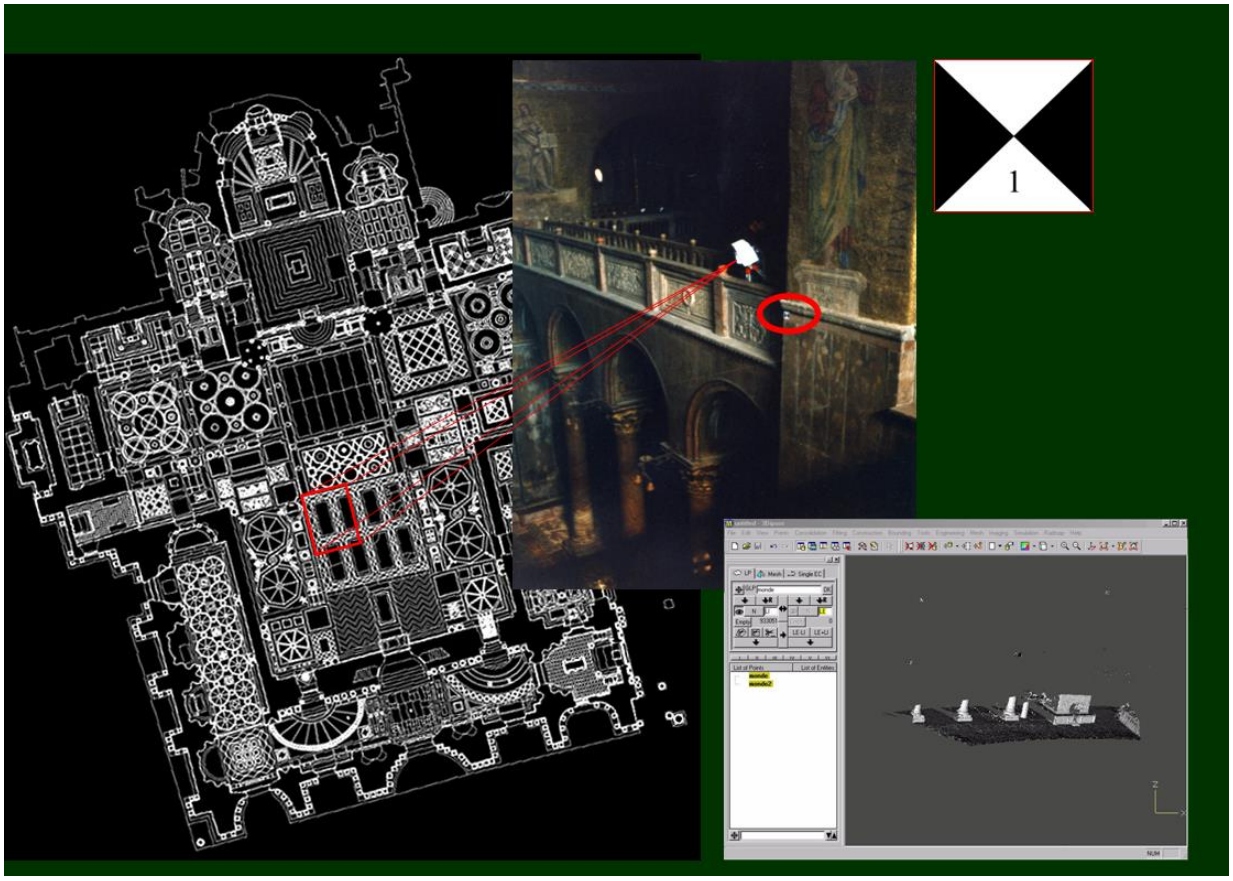
Lombardy Air Pollution Monitoring Network



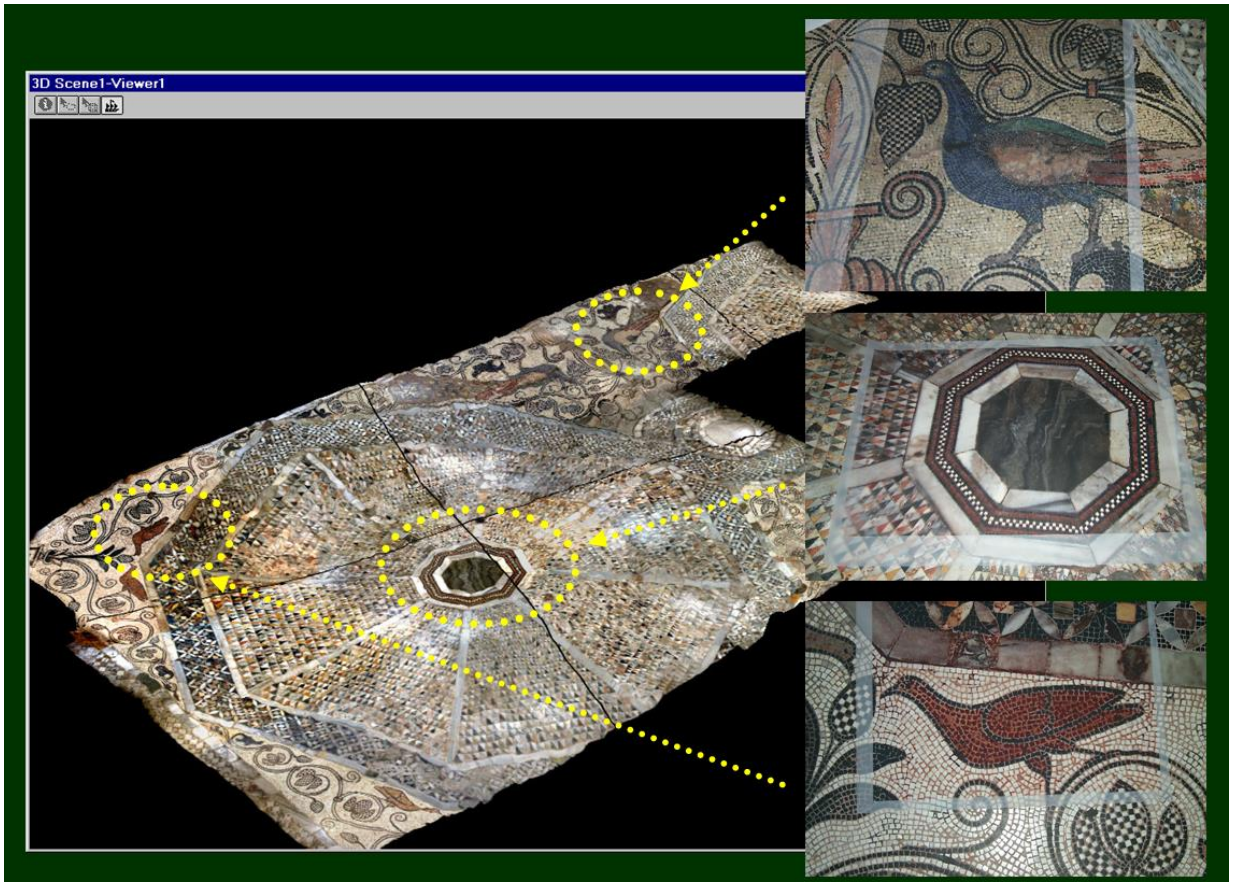
Laser-scanning of Exterior of San Lorenzo (Milan)



3D Modeling of Interior of San Lorenzo (Milan)



Photogrammetric Images of Mosaic of Saint Mark Floor (Venice)



Plotting at 1:1 scale of Mosaic of Saint Mark Floor (Venice)

AN OPENED FINAL DISCUSSION

Although in many languages the terms that indicate Power and Knowledge [17] seem to indicate a common root, no relationship is solvable, and more complex. But how? Does not it occur in the contemporary age that knowledge, as science, reaches the maximum of its power, determining not only the form of production relations, but life itself? A universal Intelligence, an Agent Intellect deployed on the entire planet, has been producing an uninterrupted revolution for some centuries, informing every aspect of our existence. And yet this formidable Science is co-conscious, and precisely in its most representative exponents, that what it produces does not continue to be in its power, continues to transform itself into the property of others. That freedom, without which Science could never have been able to achieve its formidable successes, does not in itself become a liberating energy for all of our kind. Its products, which it does not have, tend to turn into factors of enslavement and homologation. The problem then becomes that of the relationship between the knowledge and the political power. And the following questions arise: is it in the essence of knowing to turn to political power to inform it of itself? If the answer is affirmative, then knowledge would have the duty to engage politically, that is to try to hold effective power. What knowledge? The strictly scientific one? Or another kind of knowledge? The paradigm of a knowledge that is supposed to be founded in epistemic sense and which on this foundation intends to build the City, remains the Platonic one. If the city wants to be, do not be reduced to a ship on which sheep without a shepherd are boarded, or worse masked as helmsmen, it is necessary that it is imitation of the well-educated soul, that is, governed by its rational part. Science alone can unify the multiple, confer to each part its meaning and its mission, impose the superiority of the whole on the parts themselves. Nothing is more unreasonable than wanting to rationalize human vicissitudes - Leopardi will object.

And after all Aristotle had already supported him: never will the City be reducible to One; its form is a becoming from government to government and within each of them peace can only be an armistice. Politics is so called to the constitution of an order, corresponding to the same political nature of the animal man, but this order will never be that of the principles and laws that the propositions of Science can express. That of the Politician is the insecure kingdom of the most, impotent to grant the government the universal and necessary principles of knowledge. And the latter, in turn, impotent to build the city to a true image of the consistency and consistency of his speech. Thus political virtue will not be confused with the goodness of the true sage. An art of temperance and mediation is required of the politician, an art that will always remain alien to the forms and purposes of science. The reflection of the West on the Politician will be oriented towards Aristotelian realism, never forgetting the Platonic "nostalgia" ..., for the beautiful-and-good city, perfectly "in shape". This "nostalgia" is expressed in all the variations of the conception of the State as the supreme realization of individual freedom, of sovereignty as an agreement or synthesis of conflicting interests, of political society as an image of civitas in interior. Here the philosophical and scientific knowledge would still express the principles that base the security of the state. It is this knowledge only that can move the stubborn pursuit of one's private interest into that of the common Good.

Political knowledge has specialized just like any other. It concerns how to acquire the government and how to last in it. Which knowledge presupposes this art? Analysis of things as they are and not as we believe they should be; to know, therefore, the unsustainable sociability of human nature, which renders the State necessary as a coercive measure; on the basis of the regularities that emerge from the study of political cycles, to know how to prevent the dangers that run the exercise of power and to foresee its developments. This is a probabilistic and conjectural knowledge. The orders it manages to build, will always be weaker than Fortune. This is the only knowledge necessary for the housekeeper! And to this knowledge it would like to educate its which knows! But here the powerful repels it, exiles it. And yet it is a matter of knowing all reasonable within its limits, completely disenchanted. It does not induce any Magic (magic means Power); it is well known that the Prospero della Tempesta is so weak in the government as to redeem the wickedness of his fellow men. Why then does the powerful hear it? Perhaps because no knowledge can understand the irremediably double nature of power. It is not the analysis of the true actual that constitutes its essence, but the decision. The decision addressed to something that is only possible, indestructibly connected to having-being. But to "convert" to having-being, something quite different from knowledge is needed. Faith in its own ends is necessary; it is necessary to convince the listener to it. In order to build up the contingent order of the political government, it is therefore necessary to refer to an order of ideas that transcends its limit. Knowledge seems to stop in front of the intimate tragedy of political action. The pale shadow of thought, whose dimension is that of the methodical doubt, which only rational evidence solves, would arrest or delay political decision. The time of the Politician is not that of knowledge – and yet neither can they abstractly separate themselves, since both professions, that of the politician and that of the scientist, intend to discover or find possible orders in the intermixing and incessant remixing of the cases of life. Beyond any magic salvation, as well as any barren behavior similar to the Hamlet tragedy, there is dear their fraternal enmity.

[17] Free translation from Italian of the article: The art of good governance: Knowledge and Power of fraternal enemies, by Massimo Cacciari, political philosopher (La Repubblica, Wednesday 9th May 2018).

I do not know what Europe is [18]; in fact, if you saw me forced to answer with one sentence to this question, probably the most honest thing would be to take into account what St. Augustine says in his Confessions, at the beginning of a sensational reflection on the nature of time: "If nobody question what is Europe, I know, but if I want to explain it to those who ask me, I do not know ". But I'm lying: something of Europe, yes, I know it. For example, I know that for many people, perhaps especially for many young people, Europe identifies with the European Union, and that today the European Union is identified, in the worst case, with a grainy and unlikely union of countries. with so much past and little future, and, at best, with a supranational, cold, abstract and distant entity whose capital is in a cold, abstract and distant place called Brussels, which is not known with certainty what it is for except that to give work to piles of gray bureaucrats and to ensure that the populist politicians of the entire continent blame him for all that badly happens in their respective countries.

Not always, however, the image of Europe has been so negative, or at least it has not been everywhere. In reverse. For centuries Europe constituted, without going too far, the great hope of many Spaniards; aware of living from the beginning of the seventeenth century in an increasingly isolated country, increasingly immersed in poverty, in the illiteracy, in the lack of freedom, in obscurantist dogmatism and in the fiction of an empire that sank, from the middle of the eighteenth century the best my ancestors felt that Europe was a realistic promise of modernity, prosperity and freedom. I myself grew up with this idea in Spain that was trying hard to get out of the franchise. But there is no need to go back so far, nor to limit myself to my cramped experience, or that of my compatriots.

Just over a decade ago, just after the birth of the euro, while preparing the European Constitution and the enlargements of the Union and the first meetings were held to launch a common European defense, a united Europe loomed the great world power of the 21st century, the only one capable of threatening North American or Chinese domination; to the point that, in 2004, a young British political scientist like Mark Leonard ventured to publish a book entitled Why Europe will lead the 21st century and, in that same year, Jeremy Rifkin, a veteran American sociologist, could write: "While the American dream languishes, a new European dream sees the light ". And he concluded: "The Europeans have put before our eyes the vision and the path towards a new land promised for humanity". It seems impossible, but this is what very recently did Europe say thinkers of the whole world.

The question, at this point, imposes itself: what happened because all those hopes collapsed almost overnight and because, already in May 2010, an important journalist like Gideon Rachman could write in the Financial Times an article entitled The European dream is dead? The answer also imposes itself: what has happened is the deepest economic crisis that Europe has suffered since 1929, a crisis that has not triggered a world war, like the one in 1929, but a major political earthquake and the resurrection of the worst European demons, starting with the demon of nationalism, which is the demon of discord and disunity. Can Europe return to being, now that the crisis seems behind us, what it was for my Spanish ancestors, what was for me in my youth, what was for everyone or almost everyone at the beginning of this century?

Obviously, I do not know, so let's go back to the initial question: what is Europe? Does Europe have an identity like the one that apparently has Italy, Spain or Germany? And, if it has it, what does it consist of? Do they have something in common with Dante and Shakespeare, Cervantes and Montaigne, Ibsen and Kafka? Is there anything that all these writers share that do not even share a language? And by the way: just share a language to have the same identity? Do they have the same identity as Milton and Melville, Quevedo and Borges?

A few years ago, George Steiner seemed to try to define European identity in a conference entitled The Idea of Europe. He argued that our continent can be traced to five axioms.

- The first is that Europe is its cafés, those meeting places where people conspire, write and debate, and where philosophies, artistic movements, great ideological and aesthetic revolutions were born.
- The second axiom is that Europe is a tamed and practicable nature, a landscape on a human scale that contrasts with the wild and immense landscapes of Asia, America, Africa or Oceania.
- The third is that Europe is a place steeped in history, a vast lieu de la mémoire whose streets and squares are full of names that recall an ever-present past, at once bright and suffocating.
- The fourth is that Europe is the deposit of a double, contradictory and inseparable legacy: the legacy of Athens and Jerusalem, of Socrates and Jesus Christ, of reason and revelation.
- The fifth is that Europe is its own eschatological consciousness, the consciousness of its own transience, of the somber certitude that has had a beginning and will inevitably have an end, more or less tragic.

These are the five axioms that, according to Steiner, define the nature of Europe. It is almost useless to say that the idea is brilliant and provocative, but insufficient; there is no doubt that those characteristics belong to Europe, but also that they are not sufficient to define its identity. More: I'm sure Steiner knows this; and I am

[18] Free translation from Italian of the article: What happened to the European dream to turn into a nightmare?, by Javier Cercas, Spanish writer and essayist (La Repubblica – Robinson, Sunday 6th May 2018).

also sure that he knows that the problem is not the answer he gives to the question on the identity of Europe in his lecture, but in the question itself. In the second half of the sixteenth century Montaigne wrote:

> There is so much difference between us and ourselves that between us and others.

This means that, long before Freud, the great French writer understood that in a certain sense individual identity is a fiction, that inside us there is a drama and people, to use the words with which Fernando Pessoa explained the heterogeneity of his work, or that in our interior lives a confederation of souls, as claimed, inspired by Pessoa, a character of Antonio Tabucchi. Now, if individual identities are illusory, how can collective identities not be? ... In reality, the only likely European identity is precisely its diversity – a contradictory or impossible identity, an oxymoron – and the only narration capable of legitimizing it would be the narrative, more truthful, of a group of old, gifted countries. of languages, cultures, traditions and dissimilar stories that, at some point, after having spent centuries fighting ruthlessly, decide to unite to build a new and united country with the values of harmony, wellbeing and freedom of its citizens. From this point of view, the lemma of a united Europe could be one of the first lemmas of the United States, which was the great political utopia that gave birth to the Enlightenment, and historically the one that was most successful; the lemma was: *E pluribus unum*; that is: from many countries, languages, cultures, traditions and stories, only one state.

At this point I have to make a confession: for me Europe has never ceased to be what it was in my youth as a boy just emerging from an interminable dictatorship, the same that for centuries has been for the best of my Spanish ancestors; in other words: like my friend Erri De Luca, I am an extremist Europeanist. This means that, for me, united Europe is the only reasonable political utopia that we Europeans have coined. Atrocious political utopias – theoretical paradises transformed into practical infernos – we invented a lot of reasonable political utopias, which I know, only this: the utopia of a united Europe.

If I'm not mistaken, there's an infinity of obvious facts that endorse this idea; the first is that the European sport par excellence is not football, as many people believe, but the war. During the last millennium we Europeans have killed each other without giving ourselves a single month of truce and in all possible ways: in a hundred years wars, in wars of thirty years, in civil wars or religion or ethnic or world wars that in reality were basically European wars. The latter were terrible, a delirium of atrocities: as Steiner himself remembers, between August 1914 and May 1945, from Madrid to the Volga, from the Arctic to Sicily, it is estimated that a hundred million men, women and children died because of violence, hunger, deportations and ethnic cleansing, and Western Europe and the West of Russia turned into the abode of death, in the setting of unprecedented brutality, that was of Auschwitz or that of the Gulag. The project of the European Union clearly arose from the horror of that indescribable carnage and of the conviction, full of sensibility, of fatigue and of courage, that nothing like this should be repeated in Europe; the result of that conviction is not less evident, but not less surprising: my father has known the war, my grandfather, my great-grandfather and probably all my ancestors have known the war, but I do not know it; that is to say: the result is that I belong to the first generation of Europeans who do not know a war, at least – let us not forget the fierce struggles that have broken Yugoslavia – a war between the great European powers.

Of course, I know that some think that another war in Europe is now inconceivable. It seems like naivety. In the history of Europe, the rare thing is not war, but peace; moreover, it is enough to surprise serious problems again, as we have seen with the 2008 crisis, so that nationalism, which was the final cause, the ornamentation and the fuel of all the European wars of the last two, has risen in all its strength. centuries. The union of Europe was born to fight it, but it is a difficult task. Nationalism is not a political ideology: it is a faith; after all, the nation was the substitute of God as the political foundation of the state, and getting rid of it in Europe will be as difficult as it was to get rid of God. As George Orwell observed, the nationalist is indifferent to reality, so it is not important that he be demonstrated with data, for example, that getting out of Europe is a bad deal for Britain or that all the anti-immigration verbiage of Nigel Farage is none other than this, verbosity - the xenophobic delirium of a talker – because he will continue to believe that the British have to leave Europe and that immigrants threaten their work and their safety, and will therefore vote in favor of Brexit. Condorcet wrote that:

> Fear is at the origin of almost all human nonsense and, above all, political nonsense.

And Walter Benjamin argued that happiness consists in living without fear; the nationalists are unhappy with great fear: for them, for many of them, the European Union is only a distant, useless and soulless junk that forces them to live in a storm, with strange people who speak strange languages and have habits strange; they prefer to live with their own kind, or rather with those who imagine or have made them believe that they are their own, protected by the false security of all time, refugees in illusory collective identities, breathing, as Nietzsche would say, the old smell of the stable. The only way to do something useful with the future is to have the past always present, and therefore it is a huge mistake to forget the grim history of violence that

has paved Europe. To forget that the European Union was essential to erase that sinister past is an even worse mistake.
There is a second reason why the union of Europe seems to me to be the most attractive and ambitious political project of our times. We know that Europe has been for centuries the center of the world, but we also know that it is not anymore, and for some 'time this part does not pass day without hearing or reading that almost the only thing left to do to us Europeans, under the pressure of the great emerging powers, is languishing like nobles in disgrace among the ruins of our past splendor, to paraphrase the greatest Spanish poet of the post-war period: Jaime Gil de Biedma. I do not think this pessimism is justified. It is true that the weight of our countries in the world, taken one by one, is always smaller, especially if we compare it to the weight of China or India or Brazil, but it is also true that, together, we still enjoy enormous power Without going too far, we are the largest economy in the world, with a GDP of fourteen thousand billion euros. It is also true that the political weight of Europe is scarce, and also its cultural and scientific weight; but this is not due to the fact that it is united, but rather to what is not enough, that the old states resist with their claws and their teeth to yield sovereignty and to dissolve politically into a single federal state.
Utopia is still very far from being realized, and therefore nobody can be satisfied with the current functioning of the European Union: to begin with, the democratic deficit of its institutions is bloody, which is perhaps the main problem of the Union because it prevents wath initially was, by necessity, an elite project, conceived and directed by an enlightened vanguard, transformed into what must be: a popular project, directly supported and played by citizens; but here the problems begin only: we are deprived of a common economic and fiscal policy (even if not a common currency and a common bank), we do not have a common internal and foreign policy, nor a common defense policy, nor obviously a policy common cultural. From this last point of view, which is that of our little corner of readers and writers, the disunity is total, beyond the contacts and fertilizations that have always been produced and that, it is true, perhaps at this moment I am more fluids than ever; but they are totally inadequate: each of our countries operates through completely different literary, educational and intellectual systems, we do not have newspapers or magazines or common radio or television – with which we are deprived of a common public opinion – we do not have European publishing houses nor a European-wide debate, nor am I sure that we have many truly European writers, really important writers throughout the European geography.
All that I have just said may seem trivial or secondary, especially if you compare it with the great economic and political issues, but I do not think it is. Perhaps the great challenge of Europe, or of Europe in which I would like to live and on which I bet, consists precisely in reconciling two things that seem irreconcilable in principle: cultural diversity and political unity. Without cultural diversity, Europe will become irreversibly impoverished, because the variety of languages, cultures, local traditions and social autonomy is among us an almost inexhaustible source of wealth, and therefore must be looked after and strengthened. ; there is no contradiction between this urgency and that of creating a common European culture, endowed with a common intellectual system and a community of interests, because this European culture of all must be what it has always been since disintegration of the Roman Empire: the result of the fertilization of different languages and cultures. But at the same time, without political unity Europe seems condemned to destruction, because that culturally fertile diversity has been politically the seed of ethnic hatreds, of regionalist claims and of chauvinistic nationalisms that have caused the continent to clash tirelessly. threatened to annihilate him. And pluribus unam; let us return to diversity, to the multiple identity of Europe, to its original oxymoron: Europe must be politically one and culturally plural. Only in this way, it seems to me, can he give his best and not resign himself to irrelevance.

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[19] The arguments covered by the Milan School are surely larger, but these examples are certainly interesting and significant.